

GEOMETRIC PROPERTIES OF NORMED SPACES AND ESTI- MATES FOR RECTANGULAR MODULUS

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Abstract: In this paper we study some geometric properties of a normed space X using the rectangular modulus introduced in [22]. For instance, lower bounds for the rectangular modulus of $\ell^p(2)$ spaces, $p > 1$ are given. A characterization of nearly square spaces is also obtained. We prove that if X and Y are isomorphic spaces and X is close to Y (in the Banach-Mazur distance) then their rectangular moduli are also close. Using the technique of ultrapowers one obtains a sufficient condition for a normed space to have uniformly normal structure.

1. Introduction and notation

The geometric properties of a real normed space X , of dimension ≥ 2 , may be described in terms of some constants or moduli attached to X . We remember that the modulus of convexity [6], the modulus of smoothness [16], the rectangular constant [14] and the radial projection constant [26] of a normed space are often used in various applications.

Recently, K. Przeslawski and D. Yost [18] introduced the squareness modulus of the normed space X which permits to characterize uniformly smooth [20], [4], uniformly convex, uniformly non-square [4] or an inner product space [21], [4]. In [22] we have introduced the rectangular modulus of X which is intimately connected with the radial projection constant and the rectangular constant of X . Moreover, the rectangular modulus is an increasing convex function characterizing uniformly convex and inner product spaces [23], [22]. The main purpose of this paper is to continue the study of geometric properties of normed spaces using the rectangular modulus. In particular, estimates for rectangular moduli of some classes of normed spaces will be obtained.

For instance, lower bounds for the rectangular modulus of X are given in the case of two-dimensional spaces which may be identified with $B_X(x, r) = \{y \in X : \|x - y\| \leq r\}$ and $S_X(x, r) = \{y \in X : \|x - y\| = r\}$ be the closed ball, respectively the sphere with center x and radius r . Let $B_X = B_X(\theta, 1)$ be the closed unit ball of X and let $S_X = S_X(\theta, 1)$ be the unit sphere of X . The symbol \perp will be used for Birkhoff orthogonality in X . By definition $x \perp y$ if and only if $\|x\| \leq \|x + ty\|$, for all $t \in \mathbb{R}$ and geometrically this means that the line through x in the y -direction supports $B_X(\theta, \|x\|)$ at x . Recall that a Banach space X is *uniformly convex* if its *modulus of convexity* defined by

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \|x - y\| = \varepsilon \right\}, \varepsilon \in [0, 2],$$

is strictly positive for every $\varepsilon \in (0, 2]$. The Banach space X is *uniformly smooth* if its *modulus of smoothness* defined by

$$\rho_X(\tau) := \sup \left\{ \frac{1}{2} (\|x + \tau y\| + \|x - \tau y\| - 2) : x, y \in S_X \right\}, \tau \geq 0,$$

verifies the relation $\lim_{\tau \searrow 0} \rho_X(\tau)/\tau = 0$. Recall that the *Banach-Mazur distance* between two isomorphic normed spaces is the infimum of $\|T\| \|T^{-1}\|$ with respect to all isomorphisms T between them. A normed space X is said to be *nearly square* if it contains arbitrarily close copies of $\ell^1(2)$. Otherwise, X is said to be *uniformly non-square*. Any uniformly non-square space is reflexive.

For fixed $x, y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x, y)$ in the line segment $[x; y]$ with $\|z\| = 1$. Letting

$$\omega(x, y) = \frac{\|x - z(x, y)\|}{\|x\| - 1},$$

define as in [18] the *modulus of squareness* of X , by

$$\xi_X(\beta) = \sup\{\omega(x, y) : \|y\| \leq \beta < 1 < \|x\|\}, \beta \in [0, 1),$$

which is an increasing and convex function on $[0, 1)$, [4]. Recently, we have introduced the **-rectangular modulus* of X , [22], as the function $\mu_X^* : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$\mu_X^*(\lambda) = \sup\{\varphi_{\lambda, u, v}(t) : t > 0, u, v \in S_X, u \perp v\}, \lambda > 0$$

where

$$\varphi_{\lambda, u, v}(t) = \frac{\lambda + t}{\|u + tv\|}, \forall \lambda, t > 0, u, v \in S_X, u \perp v,$$

and the *rectangular modulus* of X defined by

$$\mu_X(\lambda) = \max\{\mu_X^*(\lambda), \lambda \mu_X^*(1/\lambda)\}, \lambda > 0.$$

The functions μ_X and μ_X^* are strictly increasing, convex and if X is an i.p.s. then $\mu_X(\lambda) = \mu_X^*(\lambda) = \sqrt{1 + \lambda^2}, \lambda > 0$. On the other hand $\mu_X, (\mu_X^*)$ verifies a Day-Nordlander type inequality, i.e. $\mu_X(\lambda) \geq \sqrt{1 + \lambda^2}, \forall \lambda > 0, (\mu_X^*(\lambda) \geq \sqrt{1 + \lambda^2}, \forall \lambda > 0)$. Moreover if $\mu_X(\lambda) = \sqrt{1 + \lambda^2}, (\mu_X^*(\lambda) = \sqrt{1 + \lambda^2})$ for a fixed $\lambda > 0$, then X is an inner product space [22]. In the opposite direction we have $\mu_X^*(\lambda) \leq \mu_{\ell^1(2)}(\lambda) = \lambda + 2, \forall \lambda > 0$ and consequently $\mu_X(\lambda) \leq \max\{\lambda + 2, 2\lambda + 1\}, \forall \lambda > 0, [22]$. Denoting by $\mu(X)$ the *rectangular constant* of X defined by J.L. Joly [14] and by $k(X)$ the *radial projection constant* of X defined by R.L. Thele [26], we have $\mu_X(1) = \mu_X^*(1) = \mu(X) \in [\sqrt{2}, 3]$ and $\mu_X(0+) = \mu_X^*(0+) := \lim_{\lambda \searrow 0} \mu_X^*(\lambda) = k(X) \in [1, 2], [22]$.

In [2], [3], [7], [8], [10] it is proved that, in its turn, $k(X)$ is equal to other four constants of X denoted by $MPB(X), MPB'(X), \overline{MPB}(X), \beta(X)$, respectively.

2. Main results

At the beginning we give lower bounds for the *-rectangular modulus of some particular two-dimensional spaces.

Example 1. Let X be a two-dimensional space which may be identified with \mathbb{R}^2 , with ℓ^∞ norm in the first and third quadrants. Then the unit

sphere S_X will be completed in the fourth quadrant by an arbitrary convex arc of extremities $(0, -1)$ and $(1, 0)$ and by a symmetric arc in the second quadrant. For our purpose we use the equivalent definition of μ_X^* given in [23, Lemma 2.2]:

$$\mu_X^*(\lambda) = \frac{\lambda}{\inf\{\|tu + (1-t)\lambda v\| : u, v \in S_X, u \perp v, t \in [0, 1]\}}.$$

Choosing $u_0 = (1, 0)$ and $v_0 = (0, 1)$ we obtain that

$$\begin{aligned} \mu_X^*(\lambda) &\geq \frac{\lambda}{\inf\{\|tu_0 + (1-t)\lambda v_0\| : t \in [0, 1]\}} \geq \\ &\geq \frac{\lambda}{\left\| \frac{\lambda}{1+\lambda}u_0 + \frac{\lambda}{1+\lambda}v_0 \right\|} = \lambda + 1, \forall \lambda > 0. \end{aligned}$$

This implies also that $\mu_X(\lambda) \geq \lambda + 1$.

Example 2. Let $\ell^p(2)$ be the two-dimensional ℓ^p space $1 < p < \infty$. The modulus of convexity of $\ell^p(2)$, for instance, is given explicitly for $p \geq 2$ and by an implicit formula for $p \in (1, 2)$, [13]. The modulus of squareness of $\ell^p(2)$ is not known even in the case $p = 4$, [4].

Our intention is to obtain lower bounds (asymptotically exact) for the rectangular modulus and in particular for the radial projection constant in the case of $\ell^p(2)$ spaces. Using the symmetry of $B_{\ell^p(2)}$ we can choose u (in the definition of μ_X^*) in the first quadrant and v , with $u \perp v$, in the second quadrant respectively. Then

$$\begin{cases} u = (\cos^{2/p} t, \sin^{2/p} t) = (u_1(t), u_2(t)) = (u_1, u_2), \\ v = \left(\frac{-\cos^{(2-2p)/p} t}{(\cos^{2-2p} t + \sin^{2-2p} t)^{1/p}}, \frac{\sin^{(2-2p)/p} t}{(\cos^{2-2p} t + \sin^{2-2p} t)^{1/p}} \right) = \\ = (-v_1(t), v_2(t)) = (-v_1, v_2), \end{cases}$$

for $t \in (0, \pi/2)$ and

$$\begin{aligned} \mu_{\ell^p(2)}^*(\lambda) &= \max \left\{ \frac{\lambda + s}{\|(u_1, u_2) + s(-v_1, v_2)\|_p} : s > 0, t \in (0, \pi/2) \right\} = \\ &= \max \left\{ \frac{\lambda + s}{(|u_1 - sv_1|^p + (u_2 + sv_2)^p)^{1/p}} : s > 0, t \in (0, \pi/2) \right\} = \\ &= \max\{\psi(s, t) : s > 0, t \in (0, \pi/2)\}. \end{aligned}$$

Supposing that t is fixed and $0 < s \leq u_1(t)/v_1(t)$, we conclude that for

$u_2(t)/v_2(t) \leq \lambda, \psi(s, t) \leq \psi(s_0, t)$, where

$$s_0 = \frac{u_1(u_1 + \lambda v_1)^{1/(p-1)} - u_2(\lambda v_2 - u_2)^{1/(p-1)}}{v_1(u_1 + \lambda v_1)^{1/(p-1)} + v_2(\lambda v_2 - u_2)^{1/(p-1)}} < \frac{u_1}{v_1},$$

is the root of the equation $\frac{\partial \psi}{\partial s} = 0$. In the case $0 < s \leq u_1(t)/v_1(t)$ and $u_2(t)/v_2(t) > \lambda$ we have:

$$\psi(s, t) \leq \psi\left(\frac{u_1(t)}{v_1(t)}, t\right) = \frac{\lambda v_1 + u_1}{u_1 v_2 + u_2 v_1}.$$

After a straightforward computation, one obtains that

$$\psi(s_0, t) = \frac{((\lambda v_1 + u_1)^q + (\lambda v_2 - u_2)^q)^{1/q}}{u_1 v_2 + u_2 v_1} \geq \psi\left(\frac{u_1(t)}{u_2(t)}, t\right),$$

where $q > 1$ is the conjugate of p . The case $s > u_1(t)/u_2(t)$ can be treated in a similar way. Finally it follows that

$$\psi(s, t) \leq \frac{((\lambda v_1 + u_1)^q + |\lambda v_2 - u_2|^q)^{1/q}}{u_1 v_2 + u_2 v_1}, \frac{1}{p} + \frac{1}{q} = 1,$$

and the inequality is sharp. Letting $x = \cos^2 t$ in the last inequality, after a simple calculation one obtains

$$(1) \quad \mu_{\ell^p(2)}^*(\lambda) = \max_{x \in [0,1]} \left\{ \left[\lambda(1-x)^{1/q} + x^{1/p}(x^{p/q} + (1-x)^{p/q})^{1/p} \right]^q + \left| \lambda x^{1/q} - (1-x)^{1/p}(x^{p/q} + (1-x)^{p/q})^{1/p} \right|^q \right\}^{1/q}, \frac{1}{p} + \frac{1}{q} = 1.$$

Now we observe that the radial projection constant of $\ell^p(2)$ is given by:

$$(2) \quad \begin{aligned} k(\ell^p(2)) &= \mu_{\ell^p(2)}^*(0+) = \\ &= \max_{x \in [0,1]} \left\{ (x^{q/p} + (1-x)^{q/p})^{1/q} (x^{p/q} + (1-x)^{p/q})^{1/p} \right\} = \\ &= \mu_{\ell^q(2)}^*(0+) = k(\ell^q(2)), \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

If we put in (1), $x = 0$, respectively $x = 1/2$, it follows that

$$\mu_{\ell^p(2)}^*(\lambda) \geq \max \left\{ (\lambda^q + 1)^{1/q}, \left(\frac{(\lambda + 1)^q + |\lambda - 1|^q}{2} \right)^{1/q} \right\} =$$

$$= \begin{cases} \left(\frac{(\lambda+1)^q + |\lambda-1|^q}{2} \right)^{1/q}, & q \geq 2 \\ (\lambda^q + 1)^{1/q}, & q \in (1, 2) \end{cases} = 1 + \rho_{\ell^q(2)}(\lambda), \lambda > 0,$$

where $1/p + 1/q = 1$. Remember ([19]) that the modulus of smoothness of X verifies the functional equation $\rho_X(\tau) = \tau\rho_X(1/\tau) + \tau - 1$ and that the Banach space X is uniformly convex if and only if $\lim_{\lambda \rightarrow \infty} (\mu_X^*(\lambda) - \lambda) = 0$, ([23]). Consequently

$$\begin{aligned} \mu_{\ell^p(2)}(\lambda) &= \max\{\mu_{\ell^p(2)}^*(\lambda), \lambda\mu_{\ell^p(2)}^*(1/\lambda)\} \geq \\ &\geq \max\{1 + \rho_{\ell^q(2)}(\lambda), \lambda + \lambda\rho_{\ell^q(2)}(1/\lambda)\} = 1 + \rho_{\ell^q(2)}(\lambda). \end{aligned}$$

Since $\ell^p(2)$ is uniformly convex, we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} [\mu_{\ell^p(2)}(\lambda) - 1 - \rho_{\ell^q(2)}(\lambda)] &= \lim_{\lambda \rightarrow \infty} [\mu_{\ell^p(2)}(\lambda) - \lambda - \rho_{\ell^q(2)}(\lambda) + \lambda - 1] = \\ &= \lim_{\lambda \rightarrow \infty} [\mu_{\ell^p(2)}(\lambda) - \lambda - \lambda\rho_{\ell^q(2)}(1/\lambda)] = 0. \end{aligned}$$

So, for large λ , $\mu_{\ell^p(2)}(\lambda) \approx 1 + \rho_{\ell^q(2)}(\lambda)$.

On the other hand, letting $x = 1/p$ in (2) we have:

$$k(\ell^p(2)) = k(\ell^q(2)) \geq \left(\left(\frac{1}{p} \right)^{q/p} + \left(\frac{1}{q} \right)^{q/p} \right)^{1/q} \left(\left(\frac{1}{p} \right)^{p/q} + \left(\frac{1}{q} \right)^{p/q} \right)^{1/p} = f(p)$$

and $\lim_{p \rightarrow \infty} f(p) = \lim_{p \searrow 1} f(p) = 2$. Since $k(X) = 2$ if and only if X is nearly square [25], taking into account that $k(\ell^2(2)) = 1$ it follows $\{k(\ell^p(2)) : p > 1\} = [1, 2)$. For instance, in the particular case $p = 3$ we have $k(\ell^3(2)) = (1/3)(7\sqrt{7}+17)^{1/3} \approx 1.09573$ and $f(3) = (1/3)(10\sqrt{2} + 15)^{1/3} \approx 1.02577$.

Theorem 1. *The normed space X is nearly square if and only if*

$$\lim_{\lambda \rightarrow \infty} (\mu_X^*(\lambda) - \lambda) = 2.$$

Proof. In [23, Th. 2.6] we have obtained the following estimates for μ_X^* in terms of the squareness modulus:

$$\begin{aligned} (1 - \beta(\lambda))\xi_X(\beta(\lambda)) - 1 + \beta(\lambda) &\leq \frac{\lambda}{\mu_X^*(\lambda)} (\mu_X^*(\lambda) - \lambda) \leq \\ &\leq (1 - \gamma(\lambda))\xi_X(\gamma(\lambda)) + 1 - \gamma(\lambda) + \frac{1}{4\lambda}, \forall \lambda > 2, \end{aligned}$$

where $\beta(\lambda) = \xi_X^{-1}(\lambda + 1)$, $\gamma(\lambda) = \xi_X^{-1}(\lambda - 1)$. By [4, Th. 2.7] we have

$\xi_X^{-1}(\lambda) = 1/\xi_{X^*}(1/\lambda), \forall \lambda > 1$ and this implies that

$$\begin{aligned} & \frac{\xi_{X^*}(1/(\lambda+1)) - 1}{1/(\lambda+1)} \cdot \frac{1}{\xi_{X^*}(1/(\lambda+1))} - \frac{\xi_{X^*}(1/(\lambda+1)) - 1}{\xi_{X^*}(1/(\lambda+1))} \leq \\ & \leq \frac{\lambda}{\mu_{X^*}^*(\lambda)} (\mu_{X^*}^*(\lambda) - \lambda) \leq \\ & \leq \frac{\xi_{X^*}(1/(\lambda-1)) - 1}{1/(\lambda-1)} \cdot \frac{1}{\xi_{X^*}(1/(\lambda-1))} + \frac{\xi_{X^*}(1/(\lambda-1)) - 1}{\xi_{X^*}(1/(\lambda-1))} + \frac{1}{4\lambda}. \end{aligned}$$

From $\lim_{\lambda \rightarrow \infty} \lambda/\mu_{X^*}^*(\lambda) = 1$ and $\lim_{\beta \searrow 0} \xi_X(\beta) = \xi_X(0) = 1$ we obtain

$$\lim_{\lambda \rightarrow \infty} (\mu_{X^*}^*(\lambda) - \lambda) = \xi'_{X^*}(0).$$

Since X is nearly square if and only if X^* is so (see [5, p.173]) and since by [4, Th. 2.4 (iii)], X is nearly square if and only if $\xi'_{X^*}(0) = 2$, the result follows. \diamond

Remarks. a) The normed space X is uniformly non-square if and only if

$$\lim_{\lambda \rightarrow \infty} (\mu_{X^*}^*(\lambda) - \lambda) < 2.$$

b) By $\xi'_X(0) = 2\rho'_X(0)$, [4, Theorem 2.4 (iv)] we have also $\lim_{\lambda \rightarrow \infty} (\mu_{X^*}^*(\lambda) - \lambda) = 2\rho'_{X^*}(0)$ and

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} [\mu_{X^*}^*(\lambda) - 1 - \rho_{X^*}(\lambda)] &= \lim_{\lambda \rightarrow \infty} (\mu_{X^*}^*(\lambda) - \lambda) - \lim_{\lambda \rightarrow \infty} (\lambda\rho_{X^*}(1/\lambda)) = \\ &= \rho'_{X^*}(0), \end{aligned}$$

for any normed space X . In particular X is uniformly convex if and only if $\lim_{\lambda \rightarrow \infty} [\mu_{X^*}^*(\lambda) - 1 - \rho_{X^*}(\lambda)] = 0$.

C. Benitez, K.Przeslawski and D. Yost [4] proved that for each $\beta \in (0, 1)$ the real-valued map defined on the Banach-Mazur compactum by $X \rightarrow \xi_X(\beta)$ is continuous. The core of the proof was the following special case of the Bishop-Phelps Theorem for finite dimensional spaces.

Lemma 2 [4]. *Let X be a finite dimensional normed space, $z \in X, f \in X^*, \varepsilon > 0$ with $\|z\| \leq 1, f(z) > \|f\| - \varepsilon$. Then for any $\lambda > 0$, we can find $z' \in X, f' \in X^*$ with $\|z'\| \leq 1, f'(z') = \|f'\|, \|z - z'\| \leq \varepsilon/\lambda$, and $\|f - f'\| \leq \lambda$.*

Using again Lemma 2 we shall show that for each $\lambda > 0$ the real map $X \rightarrow \mu_{X^*}^*(\lambda)$ is also continuous on the Banach-Mazur compactum. More exactly we have:

Theorem 3. *Let X and Y be two isomorphic normed spaces. Suppose that the Banach-Mazur distance between them is less than $1+2\delta^2$, where $0 < \delta < 1/7$. Then for each $\lambda > 0$*

$$|\mu_X^*(\lambda) - \mu_Y^*(\lambda)| \leq \frac{(13\lambda + 3)(2 + \lambda)^2}{\lambda} \delta.$$

Proof. Since the $*$ -rectangular modulus of a normed space is the supremum of the $*$ -rectangular moduli of its two-dimensional subspaces, we consider only the case when X and Y are finite dimensional. By hypothesis we may regard X and Y as the same vector space endowed with equivalent norms $\|\cdot\|$ and $\|\|\cdot\|\|$ such that $(1 + \delta^2)^{-1}\|\|x\|\| \leq \|x\| \leq (1 + \delta^2)\|\|x\|\|, \forall x \in X$. Choose two $\|\cdot\|$ -unit vectors u and v such that $u \perp v$ in $(X, \|\cdot\|)$. By [1, p.33] $u \perp v$ if and only if there exists $f \in X^*$ with $\|f\| = 1$ such that $f(u) = 1$ and $f(v) = 0$. Fixing such an f , we have $1 - \delta^2 \leq \|\|f\|\| \leq 1 + \delta^2, \|\|u/(1 + \delta^2)\|\| \leq 1$ and $f(u/(1 + \delta^2)) = 1/(1 + \delta^2) \geq 1 - \delta^2 \geq \|\|f\|\| - 2\delta^2$. By Lemma 2, (applied for $\varepsilon = 2\delta^2$ and $\lambda = \delta$), there exist $u' \in X, \|\|u'\|\| = 1$ and $f' \in X^*$ such that $\|\|f'\|\| = f'(u'), \|\|u' - u/(1 + \delta^2)\|\| \leq 2\delta$ and $\|\|f - f'\|\| \leq \delta$. Since

$$\|\|f'\|\| \leq \|\|f\|\| + \|\|f - f'\|\| \leq 1 + \delta + \delta^2$$

and analogously $\|\|f'\|\| \geq \|\|f\|\| - \|\|f - f'\|\| \geq 1 - \delta - \delta^2$ we have $\|\|f'\|\| = \delta_1 \in (1 - \delta - \delta^2, 1 + \delta + \delta^2)$. Let $f'' = f'/\delta_1$. Then $\|\|f''\|\| = 1, f''(u') = f'(u')/\delta_1 = 1$ and

$$\|\|f - f''\|\| \leq \|\|f - f'\|\| + \left\| \left\| \frac{\delta_1 - 1}{\delta_1} f' \right\| \right\| \leq \delta + \delta + \delta^2 < 3\delta, 0 < \delta < \frac{1}{7}.$$

Since

$$|f''(v)| = |f(v) - f''(v)| \leq \|\|f - f''\|\| \cdot \|\|v\|\| \leq 3\delta(1 + \delta^2),$$

we obtain $f''(v) = \delta_2 \in (-3\delta(1 + \delta^2), 3\delta(1 + \delta^2))$. On the other hand

$$\|\|v - \delta_2 u'\|\| \geq \|\|v\|\| - \delta_2 \|\|u'\|\| \geq 1 - \delta^2 - 3\delta(1 + \delta^2) > \frac{1}{2}, 0 < \delta < \frac{1}{7},$$

and

$$-\delta^2 - 3\delta(1 + \delta^2) \leq \|\|v - \delta_2 u'\|\| - 1 \leq \delta^2 + 3\delta(1 + \delta^2).$$

Define the $\|\|\cdot\|\|$ -unit vector v' by: $v' = (v - \delta_2 u')/\|\|v - \delta_2 u'\|\|$. Then $f''(v') = 0$ and $f''(u') = 1, \|\|f''\|\| = 1 = \|\|u'\|\|$, i.e. $u' \perp v'$ in $(Y, \|\|\cdot\|\|)$. For a fixed $\alpha \in [0, 1]$ it follows that:

$$\begin{aligned}
 & \| \alpha u + (1 - \alpha) \lambda v \| - \| \alpha u' + (1 - \alpha) \lambda v' \| \leq \\
 & \leq (1 + \delta^2) \| \alpha u + (1 - \alpha) \lambda v \| - \| \alpha u' + (1 - \alpha) \lambda v' \| \leq \\
 & \leq \alpha \| u - u' \| + (1 - \alpha) \lambda \| v - v' \| + \delta^2 \| \alpha u + (1 - \alpha) \lambda v \| \leq \\
 & \leq \| u - u' \| + \lambda \| v - v' \| + \delta^2 (1 + \delta^2) (1 + \lambda) \leq \left\| u' - \frac{u}{1 + \delta^2} \right\| + \\
 & + \left\| u \left(1 - \frac{1}{1 + \delta^2} \right) \right\| + \lambda \left\| v - \frac{v - \delta_2 u'}{\| v - \delta_2 u' \|} \right\| + \delta^2 (1 + \delta^2) (1 + \lambda) \leq \\
 & \leq 2\delta + \delta^2 + \frac{\lambda}{\| v - \delta_2 u' \|} \cdot \| (\| v - \delta_2 u' \| - 1) v + \delta_2 u' \| + \delta^2 (1 + \delta^2) (1 + \lambda) \leq \\
 & \leq 2\delta + \delta^2 + 2\lambda (\| v - \delta_2 u' \| - 1) \cdot \\
 & \cdot (\| v \| + \delta_2 \| u' \|) + \delta^2 (1 + \delta^2) (1 + \lambda) \leq \\
 & \leq 2\delta + \delta^2 + 2\lambda (1 + \delta^2) (\delta^2 + 3\delta (1 + \delta^2)) + \\
 & + 6\lambda \delta (1 + \delta^2) + \delta^2 (1 + \delta^2) (1 + \lambda) \leq \delta (13\lambda + 3), 0 < \delta < 1/7.
 \end{aligned}$$

So for a fixed triple (α, u, v) with $\alpha \in [0, 1]$ and $u, v \in S_X, u \perp v$ in X , there exists a triple (α', u', v') with $\alpha' = \alpha, u', v' \in S_Y$ and $u' \perp v'$ in Y such that

$$\| \alpha u + (1 - \alpha) \lambda v \| - \| \alpha u' + (1 - \alpha) \lambda v' \| \leq \delta (13\lambda + 3).$$

Since

$$\mu_X^*(\lambda) = \frac{\lambda}{\min \{ \| \alpha u + (1 - \alpha) \lambda v \| : \alpha \in [0, 1], u, v \in S_X, u \perp v \}},$$

by symmetry it follows that

$$\left| \frac{\lambda}{\mu_X^*(\lambda)} - \frac{\lambda}{\mu_Y^*(\lambda)} \right| \leq \delta (13\lambda + 3),$$

and

$$| \mu_X^*(\lambda) - \mu_Y^*(\lambda) | \leq \frac{\delta (13\lambda + 3)}{\lambda} \mu_X^*(\lambda) \mu_Y^*(\lambda) \leq \frac{\delta (13\lambda + 3) (\lambda + 2)^2}{\lambda}. \quad \diamond$$

Corollary 4. *Let X and Y be two isomorphic normed spaces. Suppose that the Banach-Mazur distance between them is less than $1 + 2\delta^2$, where $0 < \delta < 1/7$. Then for each $\lambda > 0$*

$$|\mu_X(\lambda) - \mu_Y(\lambda)| \leq \delta \frac{(\lambda + 1)(25\lambda^2 + 94\lambda + 25)}{\lambda}.$$

Proof. First we have

$$|\lambda \mu_X^*(1/\lambda) - \lambda \mu_Y^*(1/\lambda)| \leq \lambda \cdot \frac{(3\lambda + 13)(2\lambda + 1)^2}{\lambda^2},$$

and by Th. 3

$$\begin{aligned} |\mu_X(\lambda) - \mu_Y(\lambda)| &= \\ &= |\max\{\mu_X^*(\lambda), \lambda \mu_X^*(1/\lambda)\} - \max\{\mu_Y^*(\lambda), \lambda \mu_Y^*(1/\lambda)\}| \leq \\ &\leq |\mu_X^*(\lambda) - \mu_Y^*(\lambda)| + |\lambda \mu_X^*(1/\lambda) - \lambda \mu_Y^*(1/\lambda)| \leq \\ &\leq \delta \left[\frac{(13\lambda + 3)(\lambda + 2)^2}{\lambda} + \frac{(13 + 3\lambda)(2\lambda + 1)^2}{\lambda} \right] = \\ &= \frac{\delta(\lambda + 1)(25\lambda^2 + 94\lambda + 25)}{\lambda}. \quad \diamond \end{aligned}$$

Corollary 5. Let $\lambda > 0$ be fixed and let X be a nearly square space. Then $\mu_X^*(\lambda) = \lambda + 2$.

Proof. Since X contains arbitrarily close copies of $\ell^1(2)$ (in the Banach-Mazur distance) and since $\mu_{\ell^1(2)}^*(\lambda) = \lambda + 2$, by Th. 3 it follows that $\lambda + 2 - \varepsilon \leq \mu_X^*(\lambda) \leq \lambda + 2$, for all $\varepsilon > 0$, and the result follows. \diamond

Let A be a non-void subset of the normed space X and $x \in X$. The *radius of A relative to x* is given by $\text{rad}(x, A) = \sup_{a \in A} \|x - a\|$, and the *radius of A* is defined by $\text{rad}(A) = \inf_{x \in A} \text{rad}(x, A)$. A normed space X is said to have *normal (w-normal) structure* if the radius of each bounded convex (weakly compact convex) subset $A \subseteq X$ with $\text{diam}(A) > 0$ is strictly less than its diameter. Clearly, if X is reflexive, then normal and w-normal structure coincide. The normed space X is said to have *uniformly normal structure* if there exists $0 < c < 1$ such that $\text{rad}(A) < c \text{diam}(A)$, for any bounded convex set A , with $\text{diam}(A) > 0$.

In [4, Prop. 2.9] the authors proved that if $\xi_X(\beta) < 1/(1 - \beta)$, even for only one $\beta \in (0, 1)$ then X has uniformly normal structure. A similar result can be proved also for $*$ -rectangular modulus.

Theorem 6. Let $\lambda > 0$ be fixed and suppose that $\mu_X^*(\lambda) < \lambda + 1$. Then X has uniformly normal structure.

Proof. At the beginning we show that X has w-normal structure. Supposing that X does not have w-normal structure, then by [11,

Lemma 2.3] there exist a sequence $(Y_n)_{n \geq 1}$ of two-dimensional subspaces of X and symmetric hexagons inscribed in S_{Y_n} with length of each side within $1/n$ of 1 and with at least four sides whose distances to S_{Y_n} are $< 1/n$. We can suppose that there exists (in the Banach-Mazur compactum of two-dimensional spaces) a space Y such that $Y = \lim_n Y_n$, with respect to the Banach-Mazur distance. Generally Y is not isometric to a subspace of X , but Y can be identified with one of spaces described in Ex. 1. Then $\mu_Y^*(\lambda) \geq \lambda + 1$. By Th. 3, $\mu_X^*(\lambda) \geq \sup_n \mu_{Y_n}^*(\lambda) = \mu_Y^*(\lambda) \geq \lambda + 1$, a contradiction. Now, if $\mu_X^*(\lambda) < \lambda + 1$, by Cor. 5, the space X is uniformly non-square so that it is reflexive. Therefore X has normal structure. Denoting now by X_U an ultrapower of X , (see [12, p.146]), since X is isometric to a subspace of X_U and every finite-dimensional subspace of X_U is almost isometric to a finite dimensional subspace of X ([12, Th. 14.2]) we have that $\mu_X^*(\lambda) = \mu_{X_U}^*(\lambda)$, $\lambda > 0$. Since every ultrapower of X has normal structure, by a similar argument to that in [12, Th. 14.3], (see also [15], [17], [24]) we conclude that X has uniformly normal structure. \diamond

Remarks. a) In the limit case $\lambda = 0$, since $\mu_X^*(0+) = k(X) \geq 1$, the condition in Th. 6 becomes $\mu_X^*(0+) = 1$. In this case X is an inner product space of dimension ≥ 3 or X is a two-dimensional space having the symmetric Birkhoff orthogonality, [9]. In all cases X has uniformly normal structure.

b) If $\lim_{\lambda \rightarrow \infty} (\mu_X^*(\lambda) - \lambda) = 2\rho'_{X^*}(0) < 1$, then X and X^* have uniformly normal structure. (See also [12, Th. 14.3]).

c) If $\mu_X(\lambda) < \lambda + 1$ then X has uniformly normal structure.

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