**Mathematica Pannonica**
**11**/2 (2000), 283–291

# CEVIANS AS SIDES OF TRIANGLES

Zvonko **Čerin**

*Kopernikova 7, 10010 Zagreb, Croatia*

**Dedicated to Professor Ludwig Reich on his 60th birthday**

*Received*: December 1999

*MSC 2000*: 51 N 20, 51 M 04; 14 A 25, 14 Q 05

*Keywords*: Triangle, cevian, Euler line, triangular, central point, isogonal conjugate.

**Abstract:** We consider the problem to determine for which points $X$ in the plane of the triangle $ABC$ will lengths of cevians of $X$ be sides of a triangle. We shall prove that points from the convex hull of only ten out of hundred and one central points from Kimberling's list have this property.

## 1. Introduction

One of the basic problems in triangle geometry is to decide when three given segments are sides of a triangle. The opening chapter of the book Recent Advances in Geometric Inequalities by Mitrinović, Pečarić, and Volenec [7] gives an extensive survey of results on this question.

The present article is looking for ways of associating to a triangle $ABC$ a point $X$ of the plane such that segments $AX_a$, $BX_b$, and $CX_c$ are always sides of a triangle, where $X_a$, $X_b$, and $X_c$ are intersections of lines $AX$, $BX$, and $CX$ with the sidelines $BC$, $CA$, and $AB$, respectively. Recall [hons] that segments $AX_a$, $BX_b$, and $CX_c$ are called *cevians* of the point $X$. Some authors use this name for lines $AX$, $BX$, and $CX$.

*E-mail address:* cerin@math.hr

With a help from a computer we can describe precisely the boundary of the region $\Phi_{Ce}$ of the plane consisting of those points whose cevians are sides of a triangle. This is a curve of order 12 which has reasonably short equation provided certain changes of variables is performed. However, it is still quite complicated and our understanding of its shape and its properties as the base triangle $ABC$ changes is rather limited.

This is the reason why we consider another question which does not concern the properties of $\Phi_{Ce}$ but it gives us some information about it. The question is motivated by the fact that the centroid $G$ of $ABC$ is always in $\Phi_{Ce}$. Indeed, segments $AG_a$, $BG_b$, and $CG_c$ are medians and it is well known that medians of any triangle are sides of a triangle (see [7, p. 20] and [4, p. 282]).

Since $G$ is just one of central points of a triangle $ABC$ listed in Table 1 of [5], our original question was to find for what natural numbers $i$ less than 102 will the central point $X_i$ of the triangle $ABC$ from the Kimberling's list have the property that cevians $AX_{ia}$, $BX_{ib}$, and $CX_{ic}$ of $X_i$ are sides of a triangle. The answer gives the following theorem.

**Theorem 1.** *From* 101 *centers $X_i$ of the triangle $ABC$ from Kimberling's Table 1, only values* 2, 8, 9, 10, 21, 69, 72, 75, 76, *and* 78 *of the index $i$ have the property that the cevians $AX_{ia}$, $BX_{ib}$, and $CX_{ic}$ of $X_i$ are sides of a triangle regardless of the shape of $ABC$.*

However, we shall prove a much better result that the convex hull of the ten central points from Th. 1 consists only of points whose cevians are sides of a triangle. In other words, their convex hull always lies in $\Phi_{Ce}$. A finer analysis shows that only six points are important while the other four ($X_8$, $X_9$, $X_{10}$, and $X_{75}$) span a convex subset.

With the power of computers at our disposal, we can now consider and open up new areas of research in geometry of triangles (see [1] and [8]). This paper is simply an example of such a computer aided discovery in mathematics (see [2] and [6]).

## 2. Preliminaries

For an expression $f$, let $[f]$ denote a triple $(f, \varphi(f), \psi(f))$, where $\varphi(f)$ and $\psi(f)$ are cyclic permutations of $f$. For example, if $f = \sin A$ and $g = b + c$, then $[f]$ and $[g]$ are triples $(\sin A, \sin B, \sin C)$ and

$(b+c,\, c+a,\, a+b)$.

A triple $[a]$ of real numbers is *triangular* when $a$, $b$, and $c$ are sides of a triangle. The letter $\Omega$ is reserved for the set of all triangular triples. Let $T$ be a function that maps each triple $[a]$ of real numbers to a number $2\,a^2\,b^2 + 2\,a^2\,c^2 + 2\,b^2\,c^2 - a^4 - b^4 - c^4$. Since $T([a]) = (a+b+c)(b+c-a)(a-b+c)(a+b-c)$, it is clear that for a triple $[a]$ of positive real numbers $[a] \in \Omega$ if and only if $T(\,[a]\,) > 0$. Notice that when it is positive $T[a]$ is equal to 16 times the square of the area of $ABC$.

## 3. Placement of $ABC$

We shall position the triangle $ABC$ in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex $A$ is the origin with coordinates $(0, 0)$, the vertex $B$ is on the $x$-axis and has coordinates $(r\,h,\, 0)$, and the vertex $C$ has coordinates $(g\,q\,r/k,\, 2\,f\,g\,r/k)$, where $h = f+g$, $k = f\,g-1$, $p = f^2+1$, $q = f^2-1$, $s = g^2+1$, $t = g^2-1$, $u = f^4+1$, and $v = g^4+1$. The three parameters $r$, $f$, and $g$ are the inradius and the cotangents of half of angles at vertices $A$ and $B$. Without loss of generality, we can assume that both $f$ and $g$ are larger than 1 (i.e., that angles $A$ and $B$ are acute).

Nice features of this placement are that all central points from Table 1 in [5] have rational functions in $f$, $g$, and $r$ as coordinates and that we can easily switch from $f$, $g$, and $r$ to side lengths $a$, $b$, and $c$ and back with substitutions $c = r\,h$ and

$$a = \frac{r\,f\,s}{k}, \quad b = \frac{r\,g\,p}{k}, \quad f = \frac{(b+c)^2 - a^2}{\sqrt{T([a])}},$$

$$g = \frac{(a+c)^2 - b^2}{\sqrt{T([a])}}, \quad r = \frac{\sqrt{T([a])}}{2\,(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point $P$ with coordinates $x$ and $y$ has projections

$P_a$, $P_b$, and $P_c$ onto the side lines $BC$, $CA$, and $AB$ and $\lambda = PP_a/PP_b$ and $\mu = PP_b/PP_c$, then $x = \frac{U}{W}$ and $y = \frac{V}{W}$ with $U = g\,h\,(p\,\mu + q)\,r$, $V = 2\,f\,g\,h\,r$, and $W = f\,s\,\lambda\,\mu + g\,p\,\mu + h\,k$. This formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first trilinear coordinate. For example, we write $X_6[a]$ to indicate that the symmedian point $X_6$ has trilinears equal to $a : b : c$. Then we use the above formulas with $\lambda = a/b$ and $\mu = b/c$ to get the coordinates $x = \frac{U}{2W}$ and $y = \frac{V}{W}$ of $X_6$ in our coordinate system, where $U = (f\,q\,t + 2\,g\,u)\,g\,h\,r$, $V = f\,g\,h^2\,k\,r$, and $W = f^2\,v + f\,g\,q\,s + g^2\,u$.

## 4. Triangles from cevians

Let $x$, $y$, and $z$ be absolute trilinear coordinates of a point $X$ with respect to a base triangle $ABC$. Let $U = b\,y + c\,z$, $V = \varphi(U)$, and $W = \psi(U)$, where $a$, $b$, and $c$ are lengths of sides $BC$, $CA$, and $AB$, respectively. Let $m_a = V\,W$, $d_a = V - W$, $z_a = V + W$, $s_a = V + W - U$. Put $m_b = \varphi(m_a)$ and $m_c = \psi(m_a)$. The expressions $d_b$, $d_c$, $z_b$, $z_c$, $s_b$, and $s_c$ are defined similarly. For a natural number $k$, we let $d_{ka}$ be $V^k - W^k$. Other expressions with $k$ in their index are defined analogously.

With this notation, the cevian $AX_a$ has length $s_b\,s_c(2\,U(b^2/s_b + c^2/s_c) - a^2)/(4\,U^2)$, while cevians $BX_b$ and $CX_c$ have lengths that are its cyclic permutations. It follows that $T([\,AX_a\,])$ is the quotient $\Phi/(16\,U^4\,V^4\,W^4)$, where $\Phi$ is the cyclic sum of products $m_a(2\,b^2\,c^2\,F\,U^2 - a^4\,m_a\,G)$ and $F$ and $G$ are polynomials $\sum_{i=0}^{6} f_i\,U^i$ and $\sum_{i=0}^{6} g_i\,U^i$ with coefficients $f_0 = 4\,m_a^3\,d_a^2$, $f_1 = 2\,m_a\,z_a\,d_a^4$, $f_2 = 18\,m_a^2\,z_{2a} + 6\,m_a^3 - 5\,m_a\,z_{4a} - 2\,z_{6a}$, $f_3 = 2\,z_a^3\,(3\,z_{2a} - 7\,m_a)$, $f_4 = 8\,m_a^2 - 6\,z_{4a}$, $f_5 = 2\,z_a\,d_a^2$, $f_6 = m_a$, $g_0 = m_a^2\,d_a^4$, $g_1 = 0$, $g_2 = 2\,m_a\,d_a^2\,(d_a + V)(d_a - W)$, $g_3 = -4\,m_a\,z_a\,d_a^2$, $g_4 = 4\,z_{4a} + m_a^2 - 4\,m_a\,z_{2a}$, $g_6 = 4\,d_a^2$, and $g_5 = -4\,z_a\,(d_a + V)(d_a - W)$.

We conclude that the symmetric polynomial $\Phi$ of order 12 in variables $x$, $y$, and $z$ is the equation of the curve of order 12 which is the boundary of the region $\Omega_{Ce}$ in the plane of the base triangle $ABC$ consisting of all points whose cevians are sides of a triangle. The region $\Omega_{Ce}$ is not connected and in spite of the above rather efficient description of $\Phi$ not much could be said about its properties. The vertices of the anticomplementary triangle $A_aB_aC_a$ of $ABC$ are important singular points of this curve.

In the rest of this paper we shall be searching for central points of the triangle $ABC$ from the Kimberling's Table 1 in [5] that always belong to $\Phi_{Ce}$ regardless of the shape of $ABC$. In this direction, our main result is the following theorem.

**Theorem 2.** *For any triangle $ABC$ the convex hull $\Psi$ of central points $X_2$, $X_{21}$, $X_{69}$, $X_{72}$, $X_{76}$, and $X_{78}$ lies in $\Phi_{Ce}$, i.e., it consists only of points $X$ with the property that the cevians $AX_a$, $BX_c$, and $CX_c$ are sides of a triangle.*

**Proof.** We shall only give outlines for the proof that the vertices $X_2$ and $X_{69}$, the interior of the segment $X_2X_{69}$, and the interior of the triangle $X_2X_{21}X_{69}$ lie in the region $\Phi_{Ce}$. In a similar fashion one can show that any vertex, any segment, and any triangle on the six central points listed in the statement of the theorem have the same property.

The centroid $X_2[1/a]$ is the intersection of medians which join vertices with midpoints of opposite sides. Its coordinates are $(r\,(k\,(h+f)+f-g)/(3\,k),\,2\,r\,f\,g/(3\,k))$. Hence, $T([\,A(X_2)_a\,])$ is the quotient $9\,r^4\,g^2\,f^2\,h^2/k^2$ which is clearly always positive. Of course, this result is old and well-known (see [7, p. 20] and [4, p. 282]).

The central point $X_{69}[\cos A\,csc^2 A]$ is the isogonal conjugate of $X_{25}$ — the center of homothety of the orthic triangle $A_oB_oC_o$ and the tangential triangle $A_tB_tC_t$ of a given triangle $ABC$. It can also be described as the intersection of the line joining the Gergonne point $X_7$ with the Nagel point $X_8$ and the line joining the centroid $X_2$ with the symmedian point $X_6$.

The triangle test $T([\,A(X_{69})_a\,])$ is $48\,r^4\,S_{69}/(k^4\,p^4\,s^4)$, where $S_{69}$ is a polynomial $\sum_{i=0}^{5} k_i\,h^{2i}\,k^{\lambda_i}$, with $\lambda_i = 8, 6, 4, 2, 0, 0$ for $i = 0, \ldots, 5$ and $k_i$ is a polynomial in the variable $k$ represented as sequences $(a_0, \ldots$ $\ldots, a_n)$ of their integer coefficients as follows:

| | |
|---|---|
| $k_0$ | $-(1,\,1)^3\,(3,\,3,\,1)$ |
| $k_1$ | $-3\,(1,\,1)^2\,(2,\,1)^2(1,\,1,\,1)$ |
| $k_2$ | $-(1,\,1)\,(18,\,54,\,68,\,46,\,28,\,14,\,3)$ |
| $k_3$ | $(-12,\,-48,\,-58,\,-6,\,56,\,66,\,35,\,9,\,1)$ |
| $k_4$ | $(1,\,2,\,-2,\,-3,\,-1)(-3,\,-6,\,3,\,6,\,1)$ |
| $k_5$ | $(1,\,3,\,0,\,-5,\,0,\,3,\,1)$. |

For example, $k_0$ and $k_5$ are equal to $-(1+k)^3\,(3+3\,k+k^2)$ and $1+$ $+\,3\,k-5\,k^3+3\,k^5+k^6$, respectively.

It is not clear how one can argue that the polynomial $S_{69}$ is always positive. But, the following method will accomplish this goal.

Write $S_{69}$ in terms of $f$ and $g$. We get a polynomial $U_{69}$ with 101 terms. Since both $f$ and $g$ are larger than 1, we shall replace them with $1+f'$ and $1+g'$, where new variables $f'$ and $g'$ are positive. This substitution will give us a new polynomial $V_{69}$ with 266 terms only 12 of which have negative coefficients. If all coefficients were positive, we would be done. In order to get rid of these 12 troublesome terms, we must perform two more substitutions that reflect cases $f' \geq g'$ and $g' \geq f'$. Hence, if we replace $f'$ with $g' + \delta$ for $\delta \geq 0$, from $V_{69}$ we shall get a polynomial $P_{69}$ in $g'$ and $\delta$ with 323 terms and all coefficients positive. Similarly, if we substitute $g'$ with $f' + \varepsilon$ for $\varepsilon \geq 0$, from $V_{69}$ we shall get a polynomial $Q_{69}$ in $f'$ and $\varepsilon$ also with 323 terms and all coefficients positive. This concludes our proof that $X_{69}$ lies in $\Phi_{Ce}$.

A point $P$ in the interior of the segment $X_2X_{69}$ has coordinates $[r\,(m_1\,m_2+m_3\,x)/m_4,\ 2\,f\,g\,r\,(m_2+m_5\,x)/m_4]$, where $m_1 = f\,t+2\,g\,q$, $m_2 = v\,f^2+u\,g^2+f\,g\,q\,t$, $m_3 = 3\,f^2\,(t^3\,f+2\,g\,q\,v)$, $m_4 = 3\,k\,(x+1)\,m_2$, $m_5 = 3\,f\,g\,(h^2-k^2)$, and $x$ is a positive real number. The triangle test for cevians of this point is

$$T([AP_a]) = \frac{3\,r^4\,S_{[2,\,69]}}{k^4\,(3\,f^2\,s^2\,x+2\,m_2)^4\,(3\,g^2\,p^2\,x+2\,m_2)^4\,(3\,h^2\,k^2\,x+2\,m_2)^4}$$

where $S_{[2,\,69]}$ is a polynomial of order 12 in variable $x$ with coefficients polynomials in $f$ and $g$. In expanded form $S_{[2,\,69]}$ has 10489 terms. The replacement of $f$ with $1+f$ and $g$ with $1+g$ and then $f$ with $g+\delta$ and also $g$ with $f+\varepsilon$, where $\delta$ and $\varepsilon$ are positive, in each coefficient of powers of $x$ in $S_{[2,\,69]}$ leads to polynomials in $f$, $g$, and $\delta$ ($f$, $g$, and $\varepsilon$, respectively) with all coefficients positive which completes our proof that the interior of $X_2X_{69}$ lies in $\Phi_{Ce}$.

The Schiffler point $X_{21}[1/(\cos B + \cos C)]$ is the point of concurrence of Euler lines of triangles $BCX_1$, $CAX_1$, and $ABX_1$, where $X_1$ is the incenter of $ABC$. Recall that the line joining the centroid and the circumcenter of a scalene triangle $ABC$ is called the *Euler line* of $ABC$.

Let us now consider the triangle $X_2X_{21}X_{69}$. A point $P$ in its interior has coordinates

$$\left[\frac{r}{m_9}\,(m_0\,m_2\,m_6+m_8(m_3\,x+m_1\,m_2)),\right.$$
$$\left.\frac{2\,r}{m_9}\,(m_0\,m_2\,m_7+f\,g\,m_8(m_5\,x+m_2))\right],$$

where $m_0 = 3(s+2\,k)(x+1)\,y$, $m_6 = f\,s(q-2)+2\,g\,q$, $m_7 = q+ + 2\,k$, $m_8 = 3\,p\,s+8\,k$, $m_9 = m_2\,m_4\,m_8\,(y+1)$, and $x$ and $y$ are positive real numbers. The triangle test for the cevians of the point $P$ contains as a significant factor a polynomial of order 8 in $x$ and $y$ and whose coefficients $k_i$ $(i=0,\ldots,24)$ are polynomials in $f$ and $g$. The replacement of $f$ with $1+f$ and $g$ with $1+g$ in $k_i$ for $i=0,\ldots,24$ leads to polynomials with almost all coefficients positive. However, after we perform substitutions $f=g+u'$ and $g=f+v'$ (with $u', v' \geq 0$) we obtain polynomials with all coefficients positive which completes our proof. For other triangles the same strategy always applies but with far more complicated polynomials (with several thousands of terms and very large coefficients). $\Diamond$

**Corollary 1.** *For any triangle ABC the convex hull $\Psi_0$ of central points $X_8$, $X_9$, $X_{10}$, and $X_{75}$ lies in $\Phi_{Ce}$.*

**Proof.** By the above theorem, it suffices to prove that

(1) The Spieker centre $X_{10}[b+c/a]$ (the incenter of the triangle $A_m B_m C_m$ whose vertices are midpoints of sides) is always the interior point of the segment $X_2 X_{78}$.

(2) The Nagel point $X_8[(b+c-a)/a]$ (the intersection of lines $AA_{ea}$, $BB_{eb}$, and $CC_{ec}$, where $A_{ea}$, $B_{eb}$, and $C_{ec}$ are projections of excenters $A_e$, $B_e$, and $C_e$ onto the sidelines $BC$, $CA$, and $AB$, respectively) is always the interior point of the segment $X_{10} X_{78}$.

(3) The symmedian point of excentral triangle $X_9[b+c-a]$ (the point of concurrence of the symmedians of the excentral triangle $A_e$ $B_e$ $C_e$ also known as the Mittenpunkt) is always the interior point of the segment $X_{21} X_{78}$.

(4) The isogonal conjugate $X_{75}[1/a^2]$ of the 2nd Power Point $X_{31}[a^2]$ is always the interior point of the segment $X_{10} X_{76}$.

Since all four of these statements have the same proof we shall prove only the first.

The points $X_2$, $X_{10}$, and $X_{78}$ are collinear so that we can find a real number $x$ such that $X_{10} = (X_2 + x\,X_{78})/(x+1)$. This is the number

$$x = \frac{1}{3}\,\frac{f^2g^2 + f^2 - 4\,fg + g^2 + 5}{f^2g^2 + f^2 + 4\,fg + g^2 - 3}.$$

However, replacing $f$ with $f+1$ and $g$ with $g+1$ we get the quotient of two polynomials in $f$ and $g$ with all coefficients positive which implies that $x$ is always positive so that $X_{10}$ lies between $X_2$ and $X_{78}$. $\diamondsuit$

**Remark.**One can prove that the ten central points from the statement of our main theorem and its corollary are the only central points from Kimberling's list that always lie in $\Phi_{Ce}$. In fact, only four triangles all with $r = 1$ and

| triangle | $t_1$ | $t_2$ | $t_3$ | $t_4$ |
|---|---|---|---|---|
| $f$ | 2 | 2 | 1000 | $\frac{19}{20} + \sqrt{3}$ |
| $g$ | 5 | 20 | 1001 | $\frac{21}{20} + \sqrt{3}$ |

will suffice to eliminate the remaining 91 central points. Indeed, $T([\,A(X_i)_a\,]) \leq 0$ for the triangle $t_j$ and $i \in I_j$, where $j = 1, \ldots, 4$, $I_0 = \{\,1, \ldots, 101\,\}$, $I_3 = \{\,35, 63, 99\,\}$, $I_4 = \{\,95, 97\,\}$, $I_5 = \{\,2, 8, 9, 10, 21, 69, 72, 75, 76, 78\,\}$, $I_2 = \{5, 11, 15, 16, 30, 36, 37, 38, 45, 46, 48, 50, 52, 55, 59, 62, 68, 70, 83, 85, 86, 87, 88, 91\,\}$, and $I_1 = I_0 - I_2 - I_3 - I_4 - I_5$.

The above statement is simple to state but the reader should be aware that there is a lot of work behind it because we must know coordinates of each central point from Kimberling's list.

We can now compute the lengths of cevians of the central points $X_8$, $X_9$, $X_{10}$, $X_{21}$, $X_{75}$, $X_{76}$, and $X_{69}$, respectively, and apply the transformation formulas to get the following corollary.

**Corollary 2.** *If the triple* $[a]$ *is triangular, then the triples*

$$\left[\sqrt{\frac{2\,(b-c)^2 + a\,(b+c-a)}{a}}\right], \quad \left[\frac{\sqrt{2\,a^2\,(b^2+c^2) - a^4 - (b-c)^4}}{|(b-c)^2 - a\,(b+c)|\,\sqrt{a}}\right],$$

$$\left[\frac{\sqrt{(b+c)\,(b^2 - b\,c + c^2) + (2\,b^2 - b\,c + 2\,c^2)\;a - a^3}}{2\,a + b + c}\right],$$

$$\left[\frac{\sqrt{(2\,b^2 - b\,c + 2\,c^2)\,a^2 + a\,b\,c\,(b+c) - (b^2+c^2)\,(b-c)^2 - a^4}}{|\;(b+c)\,a^2 + 2\,a\,b\,c - (b+c)\,(b-c)^2\;|}\right],$$

$$\left[\frac{\sqrt{(b^2 - b\,c + c^2)\,(b+c)^2 - a^2\,b\,c}}{b+c}\right],$$

$$\left[\frac{\sqrt{(b^2 + c^2)\,(b^4 + c^4) - a^2\,b^2\,c^2}}{b^2 + c^2}\right],$$

*and*

$$\left[\frac{\sqrt{2\,(b^2 + c^2)\,a^2 + 3\,(b-c)^2\,(b+c)^2 - a^4}}{a}\right]$$

*are also triangular.*

## References


[1] DAVIS, P. J.: The rise, fall, and possible transfiguration of triangle geometry: A mini-history, *American Mathematical Monthly* **102** (1995), 205–214.

[2] GALE, D.: From Euclid to Descartes to Mathematica to oblivion?, *Math. Intellingencer* **14** (1992), 68–69.

[3] HONSBERGER, R.: Episodes in nineteenth and twentieth century Euclidean geometry, The Mathematical Association of America, New Mathematical Library no. 37, Washington, 1995.

[4] JOHNSON, R. A.: Advanced Euclidean Geometry, Dover Publ., New York, 1960.

[5] KIMBERLING, C.: Central points and central lines in the plane of a triangle, *Mathematics Magazine* **67** (1994), 163–187.

[6] KIMBERLING, C.: Point-to-line distances in the plane of a triangle, *Rocky Mountain Journal of Math.* **24** (1994), 1009–1026.

[7] MITRINOVIĆ, D. S. and PEČARIĆ, J. E. and VOLENEC, V.: Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, Dordrecht, 1989.

[8] OLDKNOW, A.: Computer aided research into triangle geometry, *Math. Gaz.* **80** (1996), 263–274.