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A REMARK ON CERTAIN ANALYTIC CLASS BY USING RUSCHEWEYH DERIVATIVES

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Abstract: A class $T(n, \alpha, \beta; a, b)$ of certain analytic functions defined by using Ruscheweyh derivatives, is introduced. The object of the present paper is to derive some properties of the class $T(n, \alpha, \beta; a, b)$.

1. Introduction

Let A denote the class of functions of the form

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(1.1) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. We denote by S the subclass of A consisting of functions which are univalent in U. Then a function $f(z)$ in S is said to be *starlike* of order α in U if and only if

(1.2) $$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \le \alpha < 1).$$

We denote by $S^*(\alpha)$ the class of all functions in S which are starlike of order α in U. Further, a function $f(z) \in S$ is said to be *convex* of order α in U if and only if

(1.3) $$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \le \alpha < 1).$$

We denote by $K(\alpha)$ the class of all functions in S which are convex of order α in U. It is well known that

$$S^*(\alpha) \subseteq S^*(0) = S^*$$

and

$$K(\alpha) \subset K(0) = K.$$

The classes $S^*(\alpha)$ and $K(\alpha)$ introduced by Robertson [5] were studied by Schild [8], MacGregor [3], and Pinchuk [5]. In particular, the class $S^*(\frac{1}{2})$ was studied by Schild [9] and MacGregor [3].

Recently, Ruscheweyh [7] introduced the classes K_n of functions $f(z) \in A$ satisfying

(1.4) $$\operatorname{Re}\left\{\frac{(z^n f(z))^{(n+1)}}{(z^{n-1} f(z))^{(n)}}\right\} > \frac{n+1}{2} \quad (z \in U)$$

for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $(\mathbb{N} = \{1, 2, \dots\})$. Ruscheweyh [7] showed the basic property

(1.5) $$K_{n+1} \subset K_n, \quad n \in \mathbb{N}_0.$$

We can observe that $K_0 \equiv S^*(\frac{1}{2})$ and $K_1 \equiv K$. Let

(1.6) $$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \quad n \in \mathbb{N}_0.$$

This symbol $D^n f(z)$ was introduced by Ruscheweyh [7] and was named the n-th *Ruscheweyh derivative* of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f * g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.7}$$

then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.8}$$

By using the Hadamard product, Ruscheweyh [7] observed that if

$$D^{\gamma} f(z) = \frac{z}{(a-z)^{\gamma+1}} * f(z) \quad (\gamma \geq -1) \tag{1.9}$$

then (1.6) is equivalent to (1.9) when $\gamma = n \in \mathbb{N}_0$. Thus it follows from (1.4) that the necessary and sufficient condition for $f(z) \in A$ to belong to K_n is

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{1}{2} \quad (z \in U). \tag{1.10}$$

Note that K_{-1} is the class of functions $f(z) \in A$ satisfying

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \quad (z \in U). \tag{1.11}$$

Since $K_0 \equiv S^*(\frac{1}{2})$, Ruscheweyh's result implies that $K_n \subset S^*$ for each $n \in \mathbb{N}_0$.

Let $T(n, \alpha)$ denote the class of functions satisfying the condition

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \alpha \quad (z \in U). \tag{1.12}$$

for some $0 \leq \alpha \leq \frac{1}{2}$ and $n \in \mathbb{N}_0$. The class $T(n, \alpha)$ was introduced by Goel and Sohi [2]. We observe that $T(n, \frac{1}{2}) \equiv K_n$ for each $n \in \mathbb{N}_0$. Further, Goel and Sohi [2] showed that $T(n+1, \alpha) \subset T(n, \alpha)$ for every $n \in \mathbb{N}_0$ and $0 \leq \alpha \leq \frac{1}{2}$.

Now, we define the following class $T(n, \alpha, \beta; a, b)$ by using the n-th order Ruscheweyh derivative of $f(z)$.

Definition. Let the function $f(z)$ defined by (1.1) be in the class A and

$$P(f(z); n, \alpha, \beta; a, b) = \left(\frac{D^{n+1} f(z)}{D^n f(z)} - \alpha\right)^a \left(\frac{D^{n+2} f(z)}{D^{n+1} f(z)} - \beta\right)^b, \tag{1.13}$$

where a and b are real numbers, $n \in \mathbb{N}_0$ and $\alpha, \beta \in [0, \frac{1}{2}]$.We say that

$f(z)$ belongs to the class $T(n,\alpha,\beta;a,b)$ for $n \in \mathbb{N}_0$ and $\alpha,\beta \in [0,\frac{1}{2}]$ if $f(z)$ satisfies the following condition

$$\text{Re}\left\{P(f(z);n,\alpha,\beta;a,b)\right\} > 0 \quad z \in U. \tag{1.14}$$

The powers appearing in (1.14) are meant as principle values.

We note that:

(i) $T(n,\alpha,\beta;1,0) = T(n,\alpha)$,
(ii) $T(n,\alpha,\beta;0,1) = T(n+1,\beta)$,
(iii) $T(n,\frac{1}{2},\beta;1,0) = K_n$,
(iv) $T(n,\alpha,\frac{1}{2};0,1) = K_{n+1}$,
(v) $T(n,\alpha,\alpha;a,b) = T(n,\alpha;a,b)$, was introduced in [4].

2. Results

Theorem 1. *Let $n \in \mathbb{N}_0$, and $\alpha,\beta \in [0,\frac{1}{2}]$. Then*

$$T(n,\alpha,\beta;a,b) \cap T(n,\alpha) \subset T(n,\alpha,\beta;at_1+t_2,bt_1), \tag{2.1}$$

where $|t_1| + |t_2| \le 1$.

Proof. Let the function $f(z)$ be in the class $T(n,\alpha,\beta;a,b) \cap T(n,\alpha)$ and

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha\right)^a \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \beta\right)^b = v(z). \tag{2.2}$$

Then we see that $\text{Re}\{V(z)\} > 0$ for all $z \in U$. Further let

$$\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha = u(z). \tag{2.3}$$

Then $f(z) \in T(n,\alpha)$ implies that $\text{Re}\{u(z)\} > 0$ for all $z \in U$. It follows from (2.2) and (2.3) that

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha\right)^{at_1+t_2} \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \beta\right)^{bt_1} = (u(z))^{t_2}(v(z))^{t_1}. \tag{2.4}$$

Defining the function $F(z)$ by

$$F(z) = (u(z))^{t_2}(v(z))^{t_1}, \quad |t_1| + |t_2| \le 1, \tag{2.5}$$

we obtain

$$F(0) = (1-\alpha)^{at_1+t_2}(1-\beta)^{bt_1} > 0 \tag{2.6}$$

and

$$\begin{aligned}|\arg(F(z))| &= \left|\arg\left((u(z))^{t_2}(v(z))^{t_1}\right)\right| \leq \\ &\leq |t_2|\,|\arg(u(z))| + |t_1|\,|\arg(v(z))| \leq \\ &\leq (|t_1|+|t_2|)\frac{\pi}{2} \leq \frac{\pi}{2}.\end{aligned} \tag{2.7}$$

This shows that $\mathrm{Re}\{F(z)\} > 0 (z \in U)$ which implies that $f(z) \in$ $\in T(n,\alpha,\beta;at_1+t_2,bt_1)$, $|t_1|+|t_2| \leq 1$. This completes the proof of the theorem. ◊

Putting $t_1 = t$ $t_2 = 1-t$ and $0 \leq t \leq 1$ in Th. 1, we obtain

Corollary 1. *Let $n \in \mathbb{N}_0$ and $\alpha, \beta \in [0, \frac{1}{2}]$. Then*

$$T(n,\alpha,\beta;a,b) \cap T(n,\alpha) \subset T(n,\alpha,\beta;(a-1)t+1,bt). \tag{2.8}$$

Remark. Putting $\alpha = \beta$ in Cor. 1, we have the main theorem due to Owa [4].

Putting $a = 0$ and $b = 1$ in Th. 1, we obtain:

Corollary 2. *Let $n \in \mathbb{N}_0$, $\alpha, \beta \in [0, \frac{1}{2}]$ and $|t_1|+|t_2| \leq 1$. Then we have*

$$T(n+1,\beta) \cap T(n,\alpha) \subset T(n,\alpha,\beta;t_2,t_1). \tag{2.9}$$

Theorem 2. *Let $n \in \mathbb{N}_0$, and $\alpha, \beta \in [0, \frac{1}{2}]$. Then*

$$T(n,\alpha,\beta;a,b) \cap T(n+1,\beta) \subset T(n,\alpha,\beta;at_1,bt_1+t_2), \tag{2.10}$$

where $|t_1|+|t_2| \leq 1$.

Putting $t_1 = t$, $t_2 = 1-t$ and $0 \leq t \leq 1$ in Th. 2, we obtain:

Corollary 3. *Let $n \in \mathbb{N}_0$ and $\alpha, \beta \in [0, \frac{1}{2}]$. Then*

$$T(n,\alpha,\beta;a,b) \cap T(n+1,\beta) \subset T(n,\alpha,\beta;at,(b-1)t+1), \tag{2.11}$$

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