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# ON THE IMPOSSIBILITY OF APPROXIMATING CONVEX FUNCTIONS WITH $C^2$ ONES

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**Dedicated to Professor Ludwig Reich on his 60th birthday**

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**Abstract**: We consider the problem of approximating continuous convex functions $f \in C[a,b]$ in the sup-norm by a sequence of twice continuously differentiable functions $\{f_n\}$ such that $\{\|f_n''\|\}$ is bounded. The same problem is studied for convex functions over normed spaces.

In many papers are studied the uniform approximation of a convex and continuous function by convex polynomials using the moduli of smoothness. On the other hand, it is known the equivalence between the moduli of smoothness and a certain Peetre $K$-functional (see [2]). For example, the Ditzian-Totik modulus of smoothness $\omega_\phi^2(f,t)$ is equivalent to the following $K$-functional : $K_\phi^2(f,t) = \inf_{g \in C^2[0,1]}\{\|f-g\| + + t^2\|\phi^2 g''\|\}$, where $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$ and $\|\cdot\| = \|\cdot\|_{C[0,1]}$ is the sup-norm on $[0,1]$.

Taking into consideration the above results we propose to study the following problem: is it possible to find for a continuous convex function $f \in C[a,b]$ a sequence of twice continuously differentiable convex

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functions $\{f_n\}$ such that $f_n(x) \to f(x)$ uniformly over $[a,b]$ and $\{\|f_n''\|\}$ is bounded from above?

The answer is negative to this question. More exactly we have the following theorems established not necessary for a sequence of convex functions $\{f_n\}$.

**Theorem 1.** *Let $f \in C[a,b]$ be convex. If there exists $x_0 \in (a,b)$ at which $f$ is nondifferentiable and attains at least the one sided strict local minimum over $[a,b]$, then there is no sequence $\{f_n\}$, $f_n \in C^2[a,b]$ with the following properties:*

(i) $\|f - f_n\| \to 0$,

(ii) $\{\|f_n''\|\}$ *is bounded.*

**Proof.** We distinguish two cases depending on whether $x_0$ is a strict local minimum point or not.

a) Let $x_0$ be a strict local minimum point. Furthermore, let us suppose that there exists a sequence $\{f_n\}$, $f_n \in C^2[a,b]$ with the properties (i) and (ii). Then there exists $\alpha > 0$ such that

$$\|f_n''\| \leq \alpha \tag{1}$$

for all $n$. The convexity of $f$ implies the existence of the right hand derivative $f_r'(x_0)$ and the hypothesis assures that $f_r'(x_0) > 0$. Then there exists $\delta > 0$ such that $a < x_0 - \delta < x_0 + \delta < b$ and

$$\alpha < \frac{f_r'(x_0)}{2\delta}. \tag{2}$$

Furthermore, let $\beta > 0$ such that

$$\beta < \frac{1}{2} \cdot \min\left\{f(x_0-\delta)-f(x_0),\ f(x_0+\delta)-f(x_0),\ f_r'(x_0)\delta - \frac{1}{2}\alpha\,\delta^2\right\}. \tag{3}$$

By virtue of (i) there exists $n_\beta \geq 1$ such that $\|f - f_n\| \leq \beta$ for every $n > n_\beta$. Then

$$f(x) - \beta \leq f_n(x) \leq f(x) + \beta$$

for $x \in [x_0 - \delta, x_0 + \delta]$. Hence, by (3)

$$f_n(x_0 - \delta) \geq f(x_0 - \delta) - \beta > f(x_0) + \beta \geq f_n(x_0) \tag{4}$$

and

$$f_n(x_0 + \delta) \geq f(x_0 + \delta) - \beta > f(x_0) + \beta \geq f_n(x_0). \tag{5}$$

Then (4) and (5) imply the existence of $x_1^{(n)} \in [x_0 - \delta, x_0]$ and $x_2^{(n)} \in$ $\in [x_0, x_0 + \delta]$ such that $f_n(x_1^{(n)}) = f_n(x_2^{(n)})$. Therefore there exists $\bar{x}^{(n)} \in (x_1^{(n)}, x_2^{(n)})$ such that

$$f'_n(\bar{x}^{(n)}) = 0. \tag{6}$$

Moreover, there exists $n_0 > n_\beta$ such that

$$\bar{x}^{(n_0)} \in [x_0, x_0 + \delta]. \tag{7}$$

Indeed, in the opposite case $\bar{x}^{(n)} \in (x_1^{(n)}, x_0)$ for every $n > n_\beta$. So either $f'_n(x) > 0$ or $f'_n(x) < 0$ for all $x \in [x_0, x_0 + \delta]$ and $n > n_\beta$. If there is no $n$, $n > n_\beta$ such that $f'_n(x) > 0$ for all $x \in [x_0, x_0 + {} + \delta]$ then $f'_n(x) < 0$ for all $x \in [x_0, x_0 + \delta]$ and for all $n > n_\beta$. By Lagrange's mean value theorem for every $x \in (x_0, x_0 + \delta]$ there exists $c^{(n)} = c^{(n)}(x_0, x) \in (x_0, x)$ such that $f_n(x) - f_n(x_0) = f'_n(c^n) \cdot (x - x_0)$. Then $f(x) - f_n(x) = f(x) - f_n(x_0) - f'_n(c^{(n)}) \cdot (x - x_0) > f(x) - f_n(x_0)$ or $\lim_{\, n\to\infty}(f(x) - f_n(x)) \geq \lim_{n\to\infty}(f(x) - f_n(x_0))$. By (i) we obtain

$$f(x_0) \geq f(x), \quad x \in (x_0, x_0 + \delta]. \tag{8}$$

But $x_0$ is a strict local minimum point, therefore $f(x_0) < f(x)$ for every $x \in (x_0, x_0 + \delta]$, contradiction with (8).

Thus there exists $n_1$, $n_1 > n_\beta$ such that $f'_{n_1}(x) > 0$ for all $x \in {} \in [x_0, x_0 + \delta]$. In the same way we obtain the existence of the sequence $\{n_k\}$ such that $n_1 < n_2 < n_3 < \ldots$ and

$$f'_{n_k}(x) > 0 \tag{9}$$

for every $x \in [x_0, x_0 + \delta]$ and for every $k \geq 1$. Again, by Lagrange's mean value theorem and (1) we obtain

$$|f'_{n_k}(x) - f'_{n_k}(\bar{x}^{(n_k)})| \leq \alpha\, |x - \bar{x}^{(n_k)}|$$

for all $x \in [x_0, x_0 + \delta]$. In view of (6) and (9) we have $f'_{n_k}(x) \leq {} \leq \alpha\, |x - \bar{x}^{(n_k)}|$ for all $x \in [x_0, x_0 + \delta]$. But $\bar{x}^{(n_k)} \in (x_1^{(n_k)}, x_0)$ thus $|x - \bar{x}^{(n_k)}| \leq 2\delta$. Therefore we get $f'_{n_k}(x) \leq 2\,\alpha\,\delta$ for all $x \in [x_0, x_0 + \delta]$. Hence

$$\int_{x_0}^{x} f'_{n_k}(t)dt \leq \int_{x_0}^{x} 2\,\alpha\,\delta dt$$

or $f_{n_k}(x) - f_{n_k}(x_0) \leq 2\,\alpha\,\delta(x - x_0)$. Since $\lim_{n\to\infty}(f(x) - f_n(x)) = 0$ for all $x \in [a, b]$ we get $f(x) - f(x_0) \leq 2\,\alpha\,\delta(x - x_0)$ for all $x \in [x_0, x_0 + \delta]$. In particular $f(x_0 + \delta) - f(x_0) \leq 2\,\alpha\,\delta^2$ or $[f(x_0 + \delta) - f(x_0)]/\delta \leq 2\,\alpha\,\delta$. By (2) we obtain

$$\frac{f(x_0 + \delta) - f(x_0)}{\delta} < f'_r(x_0). \tag{10}$$

On the other hand, by convexity of $f$ we get

$$\frac{f(x_0+\delta)-f(x_0)}{\delta} \geq f'_r(x_0),$$

cotradiction with (10).

In conclusion there exists $n_0, n_0 > n_\beta$ such that

$$\|f-f_{n_0}\| \leq n_\beta, \tag{11}$$

$\bar{x}^{(n_0)} \in [x_0, x_0+\delta]$ and $f'_{n_0}(\bar{x}^{(n_0)}) = 0$. By Taylor's formula, if $x \in$ $\in [x_0, x_0+\delta]$ then

$$f_{n_0}(x) = f_{n_0}(\bar{x}^{(n_0)}) + \frac{1}{1!} f'_{n_0}(\bar{x}^{(n_0)})(x-\bar{x}^{(n_0)}) + \frac{1}{2!} f''_{n_0}(\xi)(x-\bar{x}^{(n_0)})^2 =$$
$$= f_{n_0}(\bar{x}^{(n_0)}) + \frac{1}{2}\, f''_{n_0}(\xi)(x-\bar{x}^{(n_0)})^2,$$

where $\xi = \xi(x, \bar{x}^{(n_0)})$. Then by (1) and (11) we get

$$|f(x)-f_{n_0}(x)| = |f(x) - f_{n_0}(\bar{x}^{(n_0)}) - \frac{1}{2} f''_{n_0}(\xi)(x-\bar{x}^{(n_0)})^2| \geq$$
$$\geq |f(x)-f_{n_0}(\bar{x}^{(n_0)})| - \frac{1}{2}|f''_{n_0}(\xi)| \cdot (x-\bar{x}^{(n_0)})^2 \geq$$
$$\geq |f(x)-f_{n_0}(\bar{x}^{(n_0)})| - \frac{1}{2}\ \alpha\,(x-\bar{x}^{(n_0)})^2 \geq$$
$$\geq |f(x)-f(\bar{x}^{(n_0)})| - |f(\bar{x}^{(n_0)}) - f_{n_0}(\bar{x}^{(n_0)})| -$$
$$- \frac{1}{2}\alpha\,(x-\bar{x}^{(n_0)})^2 \geq$$
$$\geq |f(x)-f(\bar{x}^{(n_0)})| - \beta - \frac{1}{2}\alpha\,(x-\bar{x}^{(n_0)})^2.$$

Hence by (11) we have

$$2\beta \geq |f(x)-f(\bar{x}^{(n_0)})| - \frac{1}{2}\alpha\,(x-\bar{x}^{(n_0)})^2. \tag{12}$$

By convexity of $f$ and the condition $\bar{x}^{(n_0)} \in [x_0, x_0+\delta]$, if $x \in [x_0, \bar{x}^{(n_0)}]$ then

$$\frac{f(\bar{x}^{(n_0)})-f(x)}{\bar{x}^{(n_0)}-x} \geq f'_r(x) \geq f'_r(x_0).$$

So

$$\begin{aligned}|f(x)-f(\bar{x}^{(n_0)})| &= f(\bar{x}^{(n_0)}) - f(x) \geq f'_r(x_0)\cdot(\bar{x}^{(n_0)}-x) = \\ &= f'_r(x_0)\cdot|x-\bar{x}^{(n_0)}|.\end{aligned} \tag{13}$$

If $x \in [\bar{x}^{(n_0)}, x_0+\delta]$ then

$$(14)\quad \begin{aligned}|f(x)-f(\bar{x}^{(n_0)})| = f(x)-f(\bar{x}^{(n_0)}) \geq f'_r(\bar{x}^{(n_0)})\cdot(x-\bar{x}^{(n_0)}) \geq \\ \geq f'_r(x_0)\cdot(x-\bar{x}^{(n_0)}) = f'_r(x_0)|x-\bar{x}^{(n_0)}|.\end{aligned}$$

Plugging (13) and (14) in (12) we have

$$(15)\qquad 2\beta \geq f'_r(x_0)\cdot|x-\bar{x}^{(n_0)}| - \frac{1}{2}\alpha\,|x-\bar{x}^{(n_0)}|^2$$

for every $x \in [x_0, x_0+\delta]$. But $\bar{x}^{(n_0)} \in [x_0, x_0+\delta]$ so $|x-\bar{x}^{(n_0)}| \leq \delta$. Therefore we can define a function $g$ mapping $[0,\delta]$ into $R$ as follows: $g(t) = f'_r(x_0)t - \alpha t^2/2$. Then, by (2) we obtain $g'(t) = f'_r(x_0) - \alpha t \geq f'_r(x_0) - \alpha\delta > f'_r(x_0) - f'_r(x_0)/2 = f'_r(x_0)/2 > 0$. Thus $\max\{g(t) : 0 \leq t \leq \delta\} = g(\delta) = f'_r(x_0)\delta - \alpha\delta^2/2$ and in view of (15) we have

$$(16)\qquad f'_r(x_0)\delta - \frac{1}{2}\alpha\,\delta^2 \leq 2\ \beta\,.$$

But the choise of $\beta$ assures $2\beta < f'_r(x_0)\delta - \alpha\delta^2/2$ (see (3)), contradiction with (16). This contradiction completes the proof of the first part.

b) Now, we consider the case when $x_0$ is a one sided strict local minimum point. Then, without loss of generality, we can suppose that $f'_r(x_0) > 0$. So there exists $\delta > 0$ such that $a < x_0-\delta < x_0+\delta < b$ and

$$(17)\qquad f(x) = f(x_0)$$

for every $x \in [x_0-\delta, x_0]$ and $f(x) > f(x_0)$ for every $x \in (x_0, x_0+\delta]$. Again, let us suppose the existence of a sequence $\{f_n\}$, $f_n \in C^2[a,b]$ with the properties (i) and (ii). By (ii) there exists $\alpha > 0$ such that

$$(18)\qquad \|f''_n\| \leq \alpha$$

for every $n \geq 1$. Using Taylor's formula we obtain

$$f_n(x) = f_n(x_0) + \frac{1}{1!}\ f'_n(x_0)(x-x_0) + \frac{1}{2!}\ f''_n(\xi)(x-x_0)^2,$$

where $x \in [a,b]$ and $\xi = \xi(x_0, x; n) \in (a,b)$. By (18) we have

$$-\frac{\alpha}{2}(x-x_0)^2 \leq f_n(x) - f_n(x_0) - f'_n(x_0)(x-x_0) \leq \frac{\alpha}{2}(x-x_0)^2.$$

Hence

$$(19)\quad \begin{aligned}\frac{f_n(x)-f_n(x_0)}{x-x_0} - \frac{\alpha}{2}(x-x_0) \leq f'_n(x_0) \leq \\ \leq \frac{f_n(x)-f_n(x_0)}{x-x_0} + \frac{\alpha}{2}(x-x_0)\end{aligned}$$

for $x > x_0$ and

$$\frac{f_n(x)-f_n(x_0)}{x-x_0}+\frac{\alpha}{2}(x-x_0)\le f_n'(x_0)\le \tag{20}$$
$$\le \frac{f_n(x)-f_n(x_0)}{x-x_0}-\frac{\alpha}{2}(x-x_0)$$

for $x<x_0$, respectively. By (i), (19) and (20) we obtain

$$\frac{f(x)-f(x_0)}{x-x_0}-\frac{\alpha}{2}(x-x_0)\le \underline{\lim}_{n\to\infty} f_n'(x_0)\le \tag{21}$$
$$\le \overline{\lim}_{n\to\infty} f_n'(x_0)\le$$
$$\le \frac{f(x)-f(x_0)}{x-x_0}+\frac{\alpha}{2}(x-x_0)$$

for $x>x_0$ and the other hand

$$\frac{f(x)-f(x_0)}{x-x_0}+\frac{\alpha}{2}(x-x_0)\le \underline{\lim}_{n\to\infty} f_n'(x_0)\le \tag{22}$$
$$\le \overline{\lim}_{n\to\infty} f_n'(x_0)\le$$
$$\le \frac{f(x)-f(x_0)}{x-x_0}-\frac{\alpha}{2}(x-x_0)$$

for $x<x_0$. In view of (21), and (17), (22) we have

$$f_r'(x_0)\le \underline{\lim}_{n\to\infty} f_n'(x_0)\le \overline{\lim}_{n\to\infty} f_n'(x_0)\le f_r'(x_0)$$

and

$$0\le \underline{\lim}_{n\to\infty} f_n'(x_0)\le \overline{\lim}_{n\to\infty} f_n'(x_0)\le 0.$$

Therefore there exists $\lim_{n\to\infty} f_n'(x_0) = f_r'(x_0) > 0$ and there exists $\lim_{n\to\infty} f_n'(x_0) = 0$, respectively. This contradiction finishes the proof. ◊

In the next theorem we prove the same result without use the hypothesis that $f$ attains a one sided strict local minimum over $[a,b]$ in $x_0$:

**Theorem 2.** *Let $f\in C[a,b]$ be convex. If there exists $x_0\in(a,b)$ at which $f$ is nondifferentiable, then there is no sequence $\{f_n\}$, $f_n\in$ $\in C^2[a,b]$ with the following properties:*

(i) $\|f_n-f\|\to 0$,

(ii) $\{\|f_n''\|\}$ *is bounded.*

**Proof.** Let us suppose that there exists a sequence $\{f_n\}$, $f_n\in C^2[a,b]$ with the properties (i) and (ii). Then there exists $\alpha>0$ such that

(23) $$\|f_n''\| \le \alpha$$
for all $n \ge 1$. By Taylor's formula, if $x \in [a,b]$ then
$$f_n(x) = f_n(x_0) + \frac{1}{1!} f_n'(x_0)(x - x_0) + \frac{1}{2!} f_n''(\xi)(x - x_0)^2,$$
where $\xi = \xi(x_0, x; n) \in (a,b)$. Hence, by (23) we get

(24) $$-\frac{\alpha}{2}(x - x_0)^2 \le f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0) \le \frac{\alpha}{2}(x - x_0)^2$$
for $x \in [a,b]$. If $x > x_0$ then we obtain
$$\frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{\alpha}{2}(x - x_0) \le f_n'(x_0) \le \frac{f_n(x) - f_n(x_0)}{x - x_0} + \frac{\alpha}{2}(x - x_0).$$
Using the condition (i) we have

(25) $$\begin{aligned}\frac{f(x) - f(x_0)}{x - x_0} - \frac{\alpha}{2}(x - x_0) &\le \underline{\lim}_{n\to\infty} f_n'(x_0) \le \\ &\le \overline{\lim}_{n\to\infty} f_n'(x_0) \le \\ &\le \frac{f(x) - f(x_0)}{x - x_0} + \frac{\alpha}{2}(x - x_0)\end{aligned}$$
for $x > x_0$. But $f$ is a convex function and it is nondifferentiable in $x_0 \in (a,b)$, therefore we have

(26) $$f_l'(x_0) < f_r'(x_0).$$
By (25) we obtain the existence of the limit

(27) $$\lim_{n\to\infty} f_n'(x_0) = f_r'(x_0).$$
Again, by (24) and the condition (i) we obtain for $x < x_0$ the following
$$\begin{aligned}\frac{f(x) - f(x_0)}{x - x_0} + \frac{\alpha}{2}(x - x_0) &\le \underline{\lim}_{n\to\infty} f_n'(x_0) \le \\ &\le \overline{\lim}_{n\to\infty} f_n'(x_0) \le \\ &\le \frac{f(x) - f(x_0)}{x - x_0} - \frac{\alpha}{2}(x - x_0).\end{aligned}$$
Hence

(28) $$\lim_{n\to\infty} f_n'(x_0) = f_l'(x_0).$$
In view of (27) and (28) we have $f_r'(x_0) = f_l'(x_0)$, cotradiction with (26). ◊

**Remark 1.** Let $f \in C[a,b]$ be convex. If $f$ is differentiable on $[a,b]$, then $f'$ is an increasing function on $(a,b)$. Then there exists $f''(x)$ a.e. on $[a,b]$. Let

$$\alpha_0 = \sup\{|f''(x)| : x \in [a,b] \text{ such that there exists } f''(x)\}.$$

*If* $0 < \alpha < \alpha_0$ *then there is no sequence* $\{f_n\}$, $f_n \in C^2[a,b]$ *with the properties:*

(i) $\|f_n - f\| \to 0$,
(ii) $\|f_n''\| \le \alpha$ *for all* $n$.

Indeed, in the opposite case let $x_0 \in (a,b)$ such that there exists $f''(x_0)$. By Taylor's formula, for $x \in [a,b]$ there exists $\xi = \xi(x_0, x; n) \in$ $\in [a,b]$ such that

$$f_n(x) = f_n(x_0) + \frac{1}{1!} f_n'(x_0)(x - x_0) + \frac{1}{2!} f_n''(\xi)(x - x_0)^2.$$

By reason of the proof of Th. 2 we can state

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$

for all $x \in [a,b]$. Hence, by (i) we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^2 \cdot \lim_{n\to\infty} f_n''(\xi)$$

So

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n''(\xi) = \lim_{x\to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\frac{1}{2}(x - x_0)^2} =$$

$$= \lim_{x\to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = f''(x_0).$$

Then, by (ii) we obtain $|f''(x_0)| \le \alpha$. Therefore $\alpha_0 \le \alpha$, contradiction with the choice of $\alpha$.

The next theorem formulates the assertion of Remark 1 for a twice continuously differentiable function.

**Theorem 3.** *Let* $f \in C^2[a,b]$ *be a convex function and* $0 < \alpha <$ $< \|f''\|$. *Then there is no sequence* $\{f_n\}$, $f_n \in C^2[a,b]$ *with the following properties:*

(i) $\|f_n - f\| \to 0$,
(ii) $\|f_n''\| \le \alpha$ *for all* $n$.

**Proof.** We suppose that there exists a sequence $\{f_n\}$, $f_n \in C^2[a,b]$ with the properties (i) and (ii).

Repeating the proof of Th. 2 we can establish that

$$\lim_{n\to\infty} f_n'(x) = f(x) \tag{29}$$

for all $x \in [a,b]$.

Let $x_0, x \in [a,b]$ be arbitrary such that $x_0 < x$. In view of Taylor's formula we have

$$f(x) = f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \frac{1}{2!}f''(c)(x-x_0)$$

where $c = c(x_0, x) \in (x_0, x)$ and

$$f_n(x) = f_n(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \frac{1}{2!}f_n''(c^{(n)})(x-x_0)^2$$

where $c^{(n)} = c^{(n)}(x_0, x) \in (x_0, x)$, respectively. Then

$$f(x) - f_n(x) = f(x_0) - f_n(x_0) + $$
$$+ \frac{1}{1!}[f'(x_0) - f_n'(x_0)](x-x_0) + \frac{1}{2!}[f''(c) - f_n''(c^{(n)})](x-x_0)^2.$$

Hence, by (i) and (29) we obtain $\lim_{n\to\infty} f_n''(c^{(n)}) = f''(c)$. Thus for every subinterval $[c,d] \subseteq [a,b]$ there exist $u \in (c,d)$ and a sequence $\{u_n\}$, $u_n \in (c,d)$ for all $n$ such that

(30) $$\lim_{n\to\infty} f_n''(u_n) = f''(u).$$

This means that the set

$$S = \{u \in [a,b] : \text{ there exists a sequence } \{u_n\} \text{ such that}$$
$$u_n \in [a,b] \text{ for all } n \geq 1\text{and } \lim_{n\to\infty} f_n''(u_n) = f''(u)\}$$

is dense in $[a,b]$.

On the other hand, by (ii) we have $|f_n''(u_n)| \leq \alpha$ for every $n \geq 1$. So, in view of (30) we have

(31) $$|f''(u)| \leq \alpha$$

for every $u \in S$. If $u \in [a,b] \setminus S$ then the denseness of $S$ assures the existence of a sequence $\{s_n\}$ in $S$ such that $\lim_{n\to\infty} s_n = u$. By (31) we have $|f''(s_n)| \leq \alpha$. But $f \in C^2[a,b]$ therefore $\lim_{n\to\infty} f''(s_n) = f''(u)$. So

(32) $$|f''(u)| \leq \alpha$$

for every $u \in [a,b] \setminus S$. In view of (31) and (32) we have $|f''(u)| \leq$ $\leq \alpha$ for every $u \in [a,b]$. Thus $\|f''\| \leq \alpha$. But we have the condition $\alpha < \|f''\|$, contradiction. This completes the proof. $\Diamond$

**Remark 2.** If we consider the condition

(i′) $$f_n(x) \to f(x) \text{ for all } x \in [a,b]$$

instead of (i), then the conclusions of Ths. 2 and 3 will remain valid.

The next theorems generalize the above results:

**Theorem 4.** *Let $(E, \|\cdot\|)$ be a real normed spaces, $U \subset E$ be a nonempty, convex set and $f \in C(U, R)$ be convex. If there exists an interior point $x_0$ of $U$ at which $f$ is nondifferentiable, then there is no sequence $\{f_n\}$, $f_n \in C^2(U, R)$ with the following properties:*

(i′) $f_n(x) \to f(x)$ *for all* $x \in U$,
(ii) $\{\|f_n''(x)\|\}$ *is bounded for all* $n$ *and for all* $x \in U$.

**Proof.** Let us suppose the existence of a sequence $\{f_n\}$, $f_n \in C^2(U, R)$ with the properties (i′) and (ii). Then there exists $\alpha > 0$ with

$$\|f_n''(x)\| \leq \alpha \tag{33}$$

for every $n$ and for every $x \in U$. Using Taylor's formula we obtain

$$f_n(x) = f_n(x_0) + \frac{1}{1!}f_n'(x_0)(x - x_0) + \frac{1}{2!}f_n''(x_0 + \theta(x - x_0))(x - x_0)^2,$$

where $x \in U$ and $\theta \in (0, 1)$. Then, by (33) we get

$$|f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0)| = \frac{1}{2}|f_n''(x_0 + \theta(x - x_0))(x - x_0)^2| \leq$$
$$\leq \frac{\alpha}{2}\|x - x_0\|^2.$$

Hence

$$\begin{aligned} f_n(x) - f_n(x_0) - \frac{\alpha}{2}\|x - x_0\|^2 &\leq f_n'(x_0)(x - x_0) \leq \\ &\leq f_n(x) - f_n(x_0) + \frac{\alpha}{2}\|x - x_0\|^2 \end{aligned} \tag{34}$$

with $x \in U$.

Let $h \in E$. Then there exists $t_0 > 0$ such that $x_0 + th \in U$ for every $t \in [0, t_0]$. In view of (34) we have

$$f_n(x_0 + th) - f_n(x_0) - \frac{\alpha}{2}t^2\|h\|^2 \leq tf_n'(x_0)(h) \leq$$
$$\leq f_n(x_0 + th) - f_n(x_0) + \frac{\alpha}{2}t^2\|h\|^2$$

for every $t \in [0, t_0]$. Hence

$$\begin{aligned} \frac{f_n(x_0 + th) - f_n(x_0)}{t} - \frac{\alpha}{2}t\|h\|^2 &\leq f_n'(x_0)(h) \leq \\ &\leq \frac{f_n(x_0 + th) - f_n(x_0)}{t} + \frac{\alpha}{2}t\|h\|^2 \end{aligned} \tag{35}$$

for $t > 0$. Then, by (i′) we obtain

$$\frac{f(x_0+th)-f(x_0)}{t}-\frac{\alpha}{2}t\|h\|^2 \le \varliminf_{n\to\infty} f'_n(x_0)(h) \le$$
$$(36) \qquad \le \varlimsup_{n\to\infty} f'_n(x_0)(h) \le$$
$$\le \frac{f(x_0+th)-f(x_0)}{t}+\frac{\alpha}{2}t\|h\|^2$$

for $t > 0$.

On the other hand, because $f$ is a convex function, there exists its directional derivative $\delta f(x_0)(h) = \lim_{t\searrow 0}[f(x_0+th)-f(x_0)]/t, h \in E$. Then (36) implies the existence of the limit

$$(37) \qquad \lim_{n\to\infty} f'_n(x_0)(h) = \delta f(x_0)(h).$$

Because $f'_n(x_0)$ are linear functions, we obtain the linearity of $\delta f(x_0)$.

Now, for every $h \in E$ such that $x_0 + h$ is an interior point of $U$, we obtain the following estimations:

$$|f(x_0+h)-f(x_0)-\delta f(x_0)(h)| \le$$
$$\le |f(x_0+h)-f_n(x_0+h)|+|f_n(x_0)-f(x_0)|+$$
$$+|f_n(x_0+h)-f_n(x_0)-\delta f(x_0)(h)| \le$$
$$\le |f(x_0+h)-f_n(x_0+h)|+|f_n(x_0)-f(x_0)|+$$
$$+|f_n(x_0+h)-f_n(x_0)-f'_n(x_0)(h)|+|f'_n(x_0)(h)-\delta f(x_0)(h)|.$$

Hence, by Taylor's formula and (33) we have

$$|f(x_0+h)-f(x_0)-\delta f(x_0)(h)| \le$$
$$\le |f(x_0+h)-f_n(x_0+h)|+|f_n(x_0)-f(x_0)|+$$
$$+\frac{1}{2}|f''_n(x_0+\theta h)(h)^2|+|f'_n(x_0)(h)-\delta f(x_0)(h)| \le$$
$$\le |f(x_0+h)-f_n(x_0+h)|+|f_n(x_0)-f(x_0)|+$$
$$+\frac{\alpha}{2}\|h\|^2+|f'_n(x_0)(h)-\delta f(x_0)(h)|.$$

Using (i′) and (37) we obtain

$$\frac{1}{\|h\|}\cdot|f(x_0+h)-f(x_0)-\delta f(x_0)(h)| \le \frac{\alpha}{2}\|h\|.$$

Therefore

$$\lim_{h\to 0}\frac{1}{\|h\|}\cdot|f(x_0+h)-f(x_0)-\delta f(x_0)(h)| = 0,$$

i.e. $f$ is differentiable in $x_0$, contradiction with the hypothesis. ◊

**Theorem 5.** *Let $(E, \|\cdot\|)$ be a real normed spaces, $U \subset E$ be a nonempty, open, convex set and $f \in C^2(U,R)$ be a convex function. If $0 < \alpha < \sup\{\|f''(x)\| : x \in U\}$ then there is no sequence $\{f_n\}$, $f_n \in C^2(U,R)$ with the following properties:*

(i′) $f_n(x) \to f(x)$ *for all* $x \in U$,
(ii) $\|f_n''(x)\| \leq \alpha$ *for all* $n$ *and for all* $x \in U$.

**Proof.** The proof doesn't contain new ideas in comparison with Ths. 3 and 4, so we omit that. ◊

## References

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