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A CHARACTERIZATION OF DERIVATIONS BY FUNCTIONAL EQUATIONS

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Abstract: Let K be a field containing $\mathbb{Q}$, and let $f, h : K \to K$ be additive functions satisfying a functional equation

$$h\left(\frac{ax^n+b}{cx^n+d}\right) = \frac{x^{n-1}f(x)}{(cx^n+d)^2} \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)\,.$$

Under mild additional assumptions, it is proved that the function $F(x) = f(x) - f(1)x$ is a derivation.

In a series of papers, most of them together with Ludwig Reich, we characterized field homomorphisms and derivations among additive functions by functional equations, see [1], [2], [3], [4]. In this note, I use the methods derived in [4] to give a characterization of derivations by functional equations which arise from the differentiation of Möbius transformations.

Theorem. *Let K be a field containing $\mathbb{Q}$, $n \in \mathbb{Z} \setminus \{0\}$ and*

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

such that one of the following conditions is satisfied:

1. $c = 0,\ n \neq 1.$
2. $d = 0,\ n \neq -1.$
3. $cd \neq 0$ *and* $(c^{-1}d)^2 = e^n$ *for some* $e \in K$.

Let $f, h : K \to K$ *be additive functions, and suppose that the functional equation*

$$h\left(\frac{ax^n + b}{cx^n + d}\right) = \frac{x^{n-1} f(x)}{(cx^n + d)^2} \tag{1}$$

holds for all $x \in K^\times$ *satisfying* $cx^n + d \neq 0$. *Then the function* $F : K \to K$, *defined by*

$$F(x) = f(x) - f(1)x, \tag{2}$$

is a derivation. Moreover, we have

$$f(1) = \begin{cases} h(c^{-1}b), & \text{if} \quad d = 0, \\ h(d^{-1}a), & \text{if} \quad c = 0, \\ -e^{-1}f(e), & \text{if} \quad cd \neq 0. \end{cases} \tag{3}$$

Before we give the complete proof of the Th., we treat the special cases arising under the conditions 1. and 2.

Lemma. *Let* K *be a field containing* $\mathbb{Q}$, $n \in \mathbb{Z}\setminus\{0, 1\}$, $a \in K^\times$, $b \in K$, *and let* $f, g : K \to K$ *be additive functions satisfying the functional equation*

$$h(ax^n + b) = x^{n-1} f(x) \tag{4}$$

for all $x \in K^\times$. *Then the function* $F : K \to K$, *defined by* $F(x) = f(x) - f(1)x$, *is a derivation. Moreover, we have* $h(b) = 0$ *and* $f(1) = h(a)$.

Proof. For $x \in K^\times$ and any $t \in \mathbb{Q}^\times$, we have

$$h\big(a(tx)^n + b\big) = t^n h(ax^n) + h(b) = (tx)^{n-1} f(tx) = t^n x^{n-1} f(x)\ ,$$

and consequently

$$t^n \left[h(ax^n) - x^{n-1} f(x)\right] = -h(b)\ .$$

Since this equation holds for all $t \in \mathbb{Q}^\times$, we obtain $h(b) = 0$ and $h(ax^n) = x^{n-1} f(x)$. Now we define $g : K \to K$ by $g(x) = h(ax)$ and apply [1], Th. 2 to the pair (f, g). ◊

Proof of the Theorem. If $c = 0$ or $d = 0$, the assertions follow by the Lemma. Thus we suppose that $cd \neq 0$, and we may assume that $c = 1$. Then the decomposition

$$\frac{ax^n + b}{x^n + d} = a - \frac{D}{x^n + d}, \text{ where } D = ad - b \neq 0 ,$$

yields the functional equation

$$(5) \qquad h\left(\frac{D}{x^n + d}\right) = h(a) - \frac{x^{n-1} f(x)}{(x^n + d)^2} ,$$

valid for all $x \in K^\times$ such that $cx^n + d \neq 0$. If $x \in K^\times$ and $x^n + d \neq 0$, then $(ex^{-1})^n + d = dx^{-n}(x^n + d) \neq 0$ and

$$\frac{D}{x^n + d} = \frac{D}{d} - \frac{D}{(ex^{-1})^n + d} .$$

Applying (5) to this identity, we obtain

$$h(a) - \frac{n^{n-1} f(x)}{x^n + d)^2} = h\left(\frac{D}{d}\right) - h(a) + \frac{(ex^{-1})^{n-1} f(ex^{-1})}{((ex^{-1})^n + d)^2} ,$$

and consequently

$$(6) \qquad \frac{x^{n-1}}{(x^n + d)^2}\left[x^2 e^{-1} f(ex^{-1}) + f(x)\right] = h\left(2a + \frac{D}{d}\right)$$

for all $x \in K^\times$ satisfying $x^n + d \neq 0$. If $x \in K^\times$ and $x^n + d \neq 0$, then there are infinitely many $t \in \mathbb{Q}^\times$ such that $(tx)^n + d \neq 0$, and for these values of t we may replace x by tx in (6) and obtain

$$\frac{t^n x^{n-1}}{(t^n x^n + d)^2}\left[x^2 e^{-1} f(ex^{-1}) + f(x)\right] = h\left(2a + \frac{D}{d}\right) .$$

This is only possible, if $h(2a + \frac{D}{d}) = 0$ and $x^2 e^{-1} f(ex^{-1}) + f(x) = 0$. Hence we get the functional equation

$$(7) \qquad -e^{-1} f(ex^{-1}) = x^{-2} f(x) ,$$

valid for all $x \in K^\times$ satisfying $x^n + d \neq 0$. If $x \in K^\times$ is arbitrary, there exists some $t \in \mathbb{Q}^\times$ such that $(tx)^n + d \neq 0$. We replace x by tx in (7), divide by t and see that (7) is valid for all $x \in K^\times$. Now we define $g : K \to K$ by $g(x) = -e^{-1} f(ex)$ and obtain the functional equation

$$g(x^{-1}) = x^{-2} f(x) \quad \text{for all} \quad x \in K^\times .$$

By [1], Th. 2, the assertion follows. ◊

Corollary. *Let K be a field containing $\mathbb{Q}$, $n \in \mathbb{Z} \setminus \{0\}$ and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$$

such that one of the following conditions is satisfied:

1. $c = 0,\ n \neq 1.$
2. $d = 0,\ n \neq -1.$
3. $cd \neq 0$ *and* $(c^{-1}d)^2 = e^n$ *for some* $e \in \mathbb{Q}$.

Let $f : K \to K$ be an additive function. Then f is a derivation if and only if

$$f\left(\frac{ax^n+b}{cx^n+d}\right) = \frac{(ad-bc)nx^{n-1}f(x)}{(cx^n+d)^2} \tag{8}$$

holds for all $x \in K^\times$ such that $cx^n + d \neq 0$.

Proof. If f is a derivation, then $f \mid \mathbb{Q} = 0$ and (8) follows from the elementary properties of a derivation.

Suppose that (8) holds. By the Th., we must prove that $f(1) = 0$. If $x \in \mathbb{Q}^\times$ and $cx^n + d \neq 0$, then (8) implies

$$\frac{ax^n+b}{cx^n+d} f(1) = \frac{(ad-bc)nx^n}{(cx^n+d)^2} f(1)\,, \tag{9}$$

since f is $\mathbb{Q}$-linear. Now (9) holds for infinitely many $x \in \mathbb{Q}$, and therefore $f(1) = 0$ follows. ◊

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