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FINITARY LINEAR IMAGES OF FINITARY LINEAR GROUPS

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Abstract: A well-known and very useful result, I believe originally due to C. Chevalley, is that if G is a linear group of finite degree and if H is Zariski-closed normal subgroup of G, then G/H is isomorphic to some linear group of finite degree over the same field. We show in this paper firstly that this theorem does not extend to (infinite-dimensional) finitary linear groups and secondly in substantial special cases it does so extend.

Throughout this paper F denotes a (commutative) field and V a vector space over F. The full finitary linear group on V we denote by $FGL(V)$. Let H be a normal subgroup of the subgroup G of $FGL(V)$. We consider here circumstances which force the group G/H to be isomorphic to some finitary linear group over F.

A well-known and very useful result, I believe originally due to C. Chevalley is that if $\dim_F V$ is finite and if H is Zariski-closed, then G/H is isomorphic to a linear group over F of finite degree (see [1], p. 299 or [7] 6.4). Moreover in special cases, for example if $H = C_G(X)$ for some subset X of $\mathrm{End}_F V$ or if $H = N_G(U)$ for some subspace U of V, the degree of the representation of G/H can be bounded in terms of $\dim_F V$ only. Now the notion of Zariski topology does extend in a natural way to $FGL(V)$, for example se §25 below. Unfortunately the above theorem of Chavalley does not.

Example. a) For every characteristic $q \geq 0$ there is a completely reducible, periodic nilpotent, finitary linear group G of characteristic q with a Zariski-closed normal subgroup H such that G/H is not isomorphic to any finitary linear group.

b) For every positive characteristic p there is a unipotent nilpotent finitary linear group G of characteristic p with a Zariski-closed normal subgroup H such that G/H is not isomorphic to any finitary linear group.

In a) above H is certainly periodic with trivial unipotent radical. In [6] and [10] criteria other than Chevalley's are given for G/H to be isomorphic to a linear group of finite degree, whenever G is given as a linear group of finite degree. They all involve H being periodic with a very restricted unipotent radical and the Ex. a) above prevents any direct generalization of any of these results of [6] and [10] to finitary groups.

We are left to consider special cases of Chevalley's Theorem. Now in a linear group of finite degree, the Zariski-closure of a soluble (resp. nilpotent) subgroup is again soluble (resp. nilpotent) of the same derived length (resp. class) and this also applies to subgroups of finitary linear groups. The main result of this paper is the following.

Theorem. *Let H be a completely reducible normal subgroup of the subgroup G of $FGL(V)$. Then there is a normal subgroup $K \geq H$ of G with G/K isomorphic to a finitary linear group over F and such that H, K and F satisfy either*

a) *H and K are abelian and completely reducible, or*

b) *F is perfect and H and K are nilpotent of class c, or*

c) $\operatorname{char} F = 0$ *and H and K are soluble of derived length d.*

The following is immediate from the Theorem.

Corollary 1. *Let H be a completely reducible normal subgroup of the subgroup G of $FGL(V)$. Under any one of the following three conditions the group G/H is isomorphic to a finitary linear group over F.*

a) *H is a maximal abelian normal subgroup of G.*

b) *F is perfect and H is a maximal nilpotent-of-class-c normal subgroup of G.*

c) $\operatorname{char} F = 0$ *and H is a maximal soluble-of-derived-length-d normal subgroup of G.*

To prove the Theorem we need to consider some less specialized conditions more akin to the special cases of Chevalley's Theorem mentioned above, where the degree can be bounded. The following summarizes some of the more simply stated of these conditions.

Proposition. *Let H be a completely reducible normal subgroup of the subgroup G of $FGL(V)$ and let U be a subspace of V. Under any one of* *the following four conditions, the group G/H is isomorphic to a finitary linear group over F.*

a) $H = C_G(X)$ *for some subset* X of $\mathrm{End}_F V$.

b) $H = N_G(U)$.

c) $H = \{g \in N_G(U) \ : \ g_{|U} \text{is scalar}\}$.

d) $H = C_G(U)$.

Here a) is the basic result and b) and c) we derive from a). Part d) is very easy; we included it here for completeness. We also record some variants of a) to d); for these, as well as a proof of the Proposition, see Points **1.** to **8.** below. Parts a) and b) of the Proposition cannot be proved by following the usual proofs of Chevalley's Theorem, for example as in [1] or [7]. If one tries to do just that, at some point in the argument one arrives at an FG-module upon which G may not act finitarily. At that point the approach breaks down and one needs to attack the Proposition somewhat differently.

In the situation of both the Theorem and Proposition, I have little information if H is not completely reducible. However from Cor. 1 and the Proposition one can at least derive the following.

Corollary 2. *Let H be a normal subgroup of the subgroup G of $FGL(V)$ and suppose the unipotent radical $u(g)$ of G is trivial. If either $H = C_G(X)$ for some subset X of G, or if one of the conditions* a), b) *or* c) *of Cor. 1 hold, then G/H is isomorphic to a finitary linear group over F.*

To derive Cor. 2, note that $u(G)$ is the kernel of the action of G on the direct sum of the factors in any composition series of V as FG-module. Thus, replacing V by this direct sum, we may assume that $G \leq FGL(V)$ is completely reducible. Then H too is completely reducible by Clifford's Theorem (see [3], 5.1 or [8] Prop. 5.9). Cor. 2 now follows from Part a) of the Proposition and Cor. 1.◊

We now embark upon the proof of the Proposition and related results. For the remainder of this paper G denotes a subgroup of $FGL(V)$ and H a completely reducible normal subgroup of G. Let $V = V_0 \oplus (\oplus_{i \in I} V_i)$, where $V_0 = C_V(H)$ and the V_i for i in I are the (remaining) non-zero homogeneous components of V as FH-module. Let $E = \mathrm{End}_F V$ and $W_i = \mathrm{End}_{FH} V_i$ for each i (including $W_0 = \mathrm{End}_F V_0$). Let $W' = \oplus_{i \in I} W_i \leq C_E(H) = \mathrm{End}_{FH} V$.

1. *Let G act on W' by conjugation. Then W' is a finitary FG-module.*

Proof. We may assume that $V_0 = \{0\}$. Since H is normal in G, so G permutes the V_i. If $g \in G$ and $i, j \in I$ with $V_i g = V_j$, then $g^{-1} W_i g = W_j$. Certainly, therefore, W' is an FG-module. For any g in G, the space

$[V,g] = V(g-1)$ is finite dimensional, so $[V,g] \leq \oplus_{j\in J} V_j$ for some finite subset J of I. If $i \in I \setminus J$, then g centralizes V_i and hence g centralizes W_i. Therefore g centralizes the subspace $\oplus_{I\setminus J} W_i$ of V.

If $\dim_F V_i$ is finite, then so is $\dim_F W_i$. Suppose $\dim_F V_i$ is infinite, and suppose U is an irreducible FH-submodule of V_i. Since, by hypothesis, U is not H-trivial and V_i is FH-finitary, so V_i is a direct sum of a finite number, say r_i, of copies of U. There exists $h \in H$ with $0 < \dim_F [U,h] < \infty$. If $D = \mathrm{End}_{FH} U$, then D is a division F-algebra and $[U,h]$ is a D-submodule. Therefore $\dim_F D \leq \dim_F [U,h] < \infty$ and $\dim_F W_i = r_i^2(\dim_F D) < \infty$. Consequently $\dim_F W_i$ is finite for every i in I and hence $\dim_F (\oplus_J W_j)$ too is finite. Thus g acts finitarily on W' as claimed.◊

Remarks. In the above proof Ug is also an infinite-dimensional irreducible FH-submodule of V, so finitariness yields that $Ug \cap U \neq \{0\}$ and $Ug = U$. Thus U is an FG-submodule of V and consequently so too is V_i. Therefore W_i is normalized by G for all i in I with $\dim_F V_i$ infinite.

If $\dim_F V_0$ is infinite, then W_0 is not a finitary FG-module and V_0 has to be handled separately.

2. *Suppose $H = C_G(X)$ for some subset X of E and assume $V_0 = \{0\}$. Then $H = C_G(W')$ and the group G/H embeds into $FGL(W')$.*

Proof. Here $X \subseteq C_E(H) = \Pi_I W_i \geq W'$. If $g \in C_G(W')$, then g normalizes each W_i and so g normalizes each V_i. Hence $g \in \Pi_I \mathrm{End}_F V_i$ and, since $g \in C_G(W')$, we have

$$g \in C_G(\Pi_I W_i) \leq C_G(X) = H \quad \text{and} \quad C_G(W') \leq H.$$

Trivially $H \leq C_G(W')$. The remainder of **2.** follows.◊

3. *Suppose $H = C_G(X)$ for some subset X of E. Then G/H is isomorphic to some finitary linear group over F.*

Proof. Set $V' = \oplus_I V_i$, so $V = V_0 \oplus V'$. Set $W = V_0 \oplus W'$. Cleary **1.** implies that W is an H-trivial, finitary FG-module. Let X' be the projection of X into $\mathrm{End}_{FH} V'$ along $\mathrm{End}_F V_0$ (for X is a subset of $\mathrm{End}_{FH} V = (\mathrm{End}_{FH} V_0) \oplus (\mathrm{End}_{FH} V')$). As in the proof of **2.** we obtain $C_G(W') \leq C_G(X')$. Hence

$$C_G(W) \geq H = C_G(V_0) \cap C_G(X) =$$

$$= C_G(V_0) \cap C_G(\mathrm{End}_F V_0) \cap C_G(X') \geq C_G(V_0) \cap C_G(W') = C_G(W).$$

Therefore $H = C_G(W)$ and consequently G/H embeds into $FGL(V)$.◊

4. Corollary. *Let K be a normal subgroup of the subgroup G of $FGL(V)$. Suppose $K = N_G(Fu)$ for some $u \in V$. Then G/K is isomorphic to a finitary linear group over F.*

Proof. Let $U = FuG$. Then U is spanned by the set uG of common eigenvectors of K. In particular U is completely reducible as FK-module. Clearly

$$C_G(U) \le C_G(u) \le N_G(Fu) = K.$$

Consider the action of $G/C_G(U)$ on U. Now U has a basis $B \subseteq uG$ containing u, with respect to which K is diagonal. The full diagonal algebra X in $\mathrm{End}_F U$ with respect to B is self-centralizing, so $K \le C_G(X) \le$ $\le N_G(Fu) = K$, using that $u \in B$. The result now follows from **3.** with $U, G/C_G(U)$ and $K/C_G(U)$ playing the roles of V, G and H.◊

5. *Suppose $H = N_G(U)$ for some subspace U of V. Then G/H is isomorphic to some finitary linear group over F.*
Proof. Since H is completely reducible, so $V = U \oplus U'$ for some FH-submodule U' of V. Let $x = 1_U + 0_{U'} \in E$. Then $C_{GL(V)}(x) = GL(U) \times$ $\times GL(U')$. Hence

$$C_G(x) = N_G(U) \cap N_G(U') = N_H(U') = H.$$

The claim now follows from **3.**◊

6. *Let U be a subspace of V and suppose H=$\{g \in N_G(U) : g_{|U}$ is scalar$\}$. Then G/H is isomorphic to some finitary linear group over F.*
Proof. Again since H is completely reducible, we have $V = U \oplus U'$ as FH-module. Set $X = (\mathrm{End}_F U) + 0_{U'} \le E$. Then $C_E(X) = (F1_U) \oplus$ $\oplus(\mathrm{End}_F U')$. Hence $H \le C_G(X) \le H \cap N_G(U') = H$. Consequently $H = C_G(X)$ and **3.** yields the result.◊

7. *Let K be a normal subgroup of $G \le FGL(V)$ and U a subspace of V such that $K = C_G(U)$. Then G/H is isomorphic to some finitary linear group over F.*
Proof. Since K is normal in G, we have $UG \le C_V(K)$. Thus $K =$ $= C_G(UG)$ and G/K embeds into FGL(UG).◊

The Proposition, Parts a), b), c) and d) follows, respectively, from **3.**, **5.**, **6.** and **7.**

We can combine **5.**, **6.** and **7.** as follows.

8. *Let $\{U_j : j \in J\}$ be a family of subspaces of V and for each j in J suppose that H_j is a normal subgroup of G such that H_j is either $N_G(U_j)$ or $\{g \in N_G(U_j) : g$ is scalar on $U_j\}$ or $C_G(U_j)$. If $H = \cap_{j \in J} H_j$, then G/H is isomorphic to some finitary linear group over F.*
Proof. Replace V by $V \oplus Fv_\infty$, where v_∞ is non-zero and fixed by G. If $H_j = C_G(U_j)$, replace U_j by $U_j \oplus Fv_\infty$. Then $H_j = \{g \in N_G(U_j) :$ $: g$ is scalar on $U_j\}$. Since H is completely reducible for each j in J

there is an FH-submodule U_j' of V with $V = U_j \oplus U_j'$. If $H_j = N_G(U_j)$ set $X_j = \{1_{U_j} + 0_{U_j'}\}$. Otherwise set $X_j = (\mathrm{End}_F U_j) + 0_{U_j'}$. Then $H = C_G(\cup_{j \in J} X_j)$. Now apply **3.**◊

9. *Let N be a subgroup of $GL(n, F)$, where n is an integer, F is perfect and $\mathrm{char}\, F = p \geq 0$. If N is locally nilpotent Statements* a), b) *and* c) *below are equivalent. If N is soluble Statements* a), b) *and* d) *are equivalent.*

a) *Every cyclic subgroup of N is completely reducible.*

b) *Every subgroup of N is completely reducible.*

c) *N is completely reducible.*

d) *N is completely reducible and if $p > 0$, then the index $(N : N^0)$ is prime to p.* (Here N^0 denotes, as usual, the connected component of the identity of N in the Zariski topology.)

Proof. If N is locally nilpotent apply [7] 1.24 (or [1] p. 64) and [7] 7.7. If N is soluble apply [7] 1.24 (or [1] p. 64) and [7] 7.6.◊

10 Corollary. *Let N be a subgroup of $FGL(V)$ where F is perfect and N is a hypercentral group. Of the following conditions*, a) ⇒ b) ⇔ c) c) ⇒ d).

a) *V is completely reducible as FN-module.*

b) *V is completely $F\langle x \rangle$-reducible for every x in N.*

c) *V is completely FX-reducible for every finitely generated subgroup X of N.*

d) *$[V, N]$ is completely FN-reducible.*

Proof. a) ⇒ b). Every subgroup of N is ascendant in N. Thus here we just apply Clifford's Theorem, see [3] 5.1 or [8] Prop. 9.

b) ⇒ c). Let X be a finitely generated subgroup of N. If U is a finite-dimensional subspace of V containing $[V, X]$, then U is completely reducible as FX-module by **9.** above. Hence $U = [V, X] + C_U(X)$ for all such U and so $V = [V, X] + C_V(X)$. Therefore V is completely FX-reducible.

c) ⇒ b). This is trivial.

c) ⇒ d). By [4] or [9] the FN-module $[V, N]$ is completely reducible. ◊

We need to restrict to $[V, N]$ in **10**d); that is b) does not imply a) in **10.**

11 Example. *For any field F there is an abelian finitary linear group over F that is not completely reducible, while all its finitely generated subgroups are completely reducible.*

Proof. We modify the example in [9]. The example there is non-abelian, but with just two composition factors in the standard module. To make the group abelian we abandom the latter property.

Let F be a field with non-zero, non-identity element α. Set $W = \oplus_{1\leq i<\infty} Fv_i$, where each $v_i \neq 0$. Let $a_i \in GL(W)$ act on W by $v_i a_i = \alpha v_i$ and $v_j a_i = v_j$ for $j \neq i$. Set $G =< a_i : i \geq 1 >$. Then G is abelian, being a direct product of copies of $< \alpha >$. Let $\delta : G \to W$ be defined by $\delta : a \mapsto \sum_i (\alpha_i - 1)v_i$ for all $a \in G$, where $v_i a = \alpha_i v_i$ for each i. Almost all the α_i are 1, so $\delta : G \to W$ is well defined. Further δ is a derivation.

Set $V = Fv_0 \oplus W$, where $v_0 \neq 0$. Let $a \in G$ act on V via the matrix $\begin{pmatrix} 1 & a\delta \\ 0 & a \end{pmatrix}$; that is a maps $\beta v_0 + w$ for $\beta \in F$ and $w \in W$, to $\beta v_0 + \beta(a\delta) + wa$. Then V becomes a finitary FG-module and, since $W \leq V$, trivially V is G-faithul. W is completely FG-reducible. We claim that V is not completely FG-reducible. Since V/W is G-trivial, while no irreducible FG-submodule of W is G-trivial, the following are equivalent: V is completely FG-reducible; V splits as FG-module over W; $C_V(G) \not\leq W$; $C_V(G) \neq \{0\}$.

Suppose $v_0 + w \in C_V(G)$ for some $w \in W$. Then for any $a \in G$ we have $(v_0 + w)a = v_0 + w$, so $a\delta + wa = w$ and $a\delta = w(a-1)$. Now $w \in \oplus_{1\leq i<n} Fv_i$ for some finite n. Then $a_n\delta = w(a_n - 1) = 0$. But by definition $a_n\delta = (\alpha - 1)v_n \neq 0$. This contradiction proves that no such element $v_0 + w$ exists. Therefore V is not completely reducible as FG-module.

Let H be any finitely generated subgroup of G. Suppose α is a root of unity. Then H has finite order dividing some power of $|\alpha|$ and the latter is prime to $\operatorname{char} F$ if $\operatorname{char} F \neq 0$. Thus the group algebra FH is semisimple and so V is completely reducible as FH-module.

There remains the case where F contains no non-trivial root of unity (so $\operatorname{char} F = 2$ etc.). Suppose $|\alpha|$ above is infinite. Now $v_0 + v_1 + \cdots + v_n$ is a fixed point for $a_1, a_2, \ldots, a_n$. Then $V = F(\sum_{0\leq i\leq n} v_i) \oplus W$ as $F{<}a_1, a_2, \ldots, a_n{>}$-module. It follows that every finitely generated subgroup of G is completely reducible. There remains the case where $|F| = 2$. Let α be a primitive cube root of unity over F. Repeat the above construction with F replaced by $F(\alpha)$ and then regard V as an FG-module. If H is a finitely generated subgroup of G then $|H|$ has order a power of 3 and hence H is completely reducible. If V is completely reducible as FG-module, then $C_V(G) \neq \{0\}$ and V is completely reducible as $F(\alpha)G$-module, which it is not. The construction is complete.$\diamond$

12. *Let $x \in FGL(V)$ act diagonalizably on V. Then x acts diagonalizably via conjugation on E.* (The element x acts *diagonalizably on V* if V spanned by eigenvectors of x.)

Proof. Now $V = U \oplus C$ as $F < x >$-module, for some finite-dimensional subspace U and some subspace $C \leq C_V(x)$. Also $U = \oplus_{1 \leq i \leq n} Fu_i$ for some basis $u_1, \dots, u_n$ of U of eigenvectors of x. Let e_{ij} be the element of E defined by $u_i e_{ij} = u_j$; $u_k e_{ij} = 0$ for $k \neq i$ and $Ce_{ij} = \{0\}$. Then E as F-space is equal to

$$\big(\oplus_{ij} Fe_{ij}\big) \oplus \mathrm{End}_F C\big(\oplus_i \mathrm{Hom}_F(Fu_i, C)\big) \oplus \big(\oplus_j \mathrm{Hom}_F(C, Fu_j)\big).$$

Suppose $u_i x = \xi_i u_i$ for each i, where $\xi_i \in F$. Then $x^{-1} e_{ij} x = \xi_i^{-1} \xi_j e_{ij}$. Clearly x centralizes C. Also x acts on $\mathrm{Hom}_F(Fu_i, C)$ by multiplying by ξ_i^{-1} and acts on $\mathrm{Hom}_F(C, Fu_j)$ by multiplying by ξ_j. Thus E is spanned by eigenvectors of x and the claim follows.◊

13. *Let $x \in FGL(V)$ be such that V is completely reducible as $F<x>$-module and suppose F is perfect. Then E is completely reducible as $F<x>$-module.*

Proof. Let $F^\frown$ be the algebraic closure of F. Now $V = \oplus_{\lambda \in \Lambda} I_\lambda$ for some irreducible $F<x>$-modules I_λ. Also each $\dim_F I_\lambda$ is finite. Then [1] p. 64 or [7] 1.24 yields that $F^\frown \oplus_F V \cong \oplus_\Lambda F^\frown \oplus_F I_\lambda$ and is completely reducible as $F^\frown <x>$-module. By **12.** it follows that x acts diagonalizably on $\mathrm{End}_{F^\frown}(F^\frown \oplus_F V)$. But there is an $F^\frown <x>$ embedding of $F^\frown \oplus_F E$ into $\mathrm{End}_{F^\frown}(F^\frown \oplus_F V)$. Therefore x acts diagonalizably on $F^\frown \oplus_F E$. Consequently E is completely reducible over $F<x>$ by [1] p. 64 or [7] 1.24 again.◊

14. Consider the proof of **3.** *Suppose that F is perfect and that every element of G acts completely reducibly on V. Then every element of G acts completely reducible on W.*

Proof. Now $V = V_0 \oplus V'$ and $W = V_0 \oplus W'$, see the proof of **3**. If $x \in G$, than V is completely reducible as $F<x>$-module and hence V_0 and V' are too. Further $W' \leq \mathrm{End}_F V'$ is also completely reducible as $F<x>$-module by **13**. The claim follows.◊

15. *Consider the situation of* **1**. *Assume G is soluble and completely reducible. Then $dim_F(Fw^G)$ is finite for every w in W'.* (Here $Fw^G = \sum_{g \in G} Fw^g = \sum_{g \in G} Fg^{-1}wg$.)

Proof. Let $w \in W'$. Then $w \in \oplus_{j \in J} W_j$ for some finite subset J of I. Since G is soluble, every irreducible FG-submodule and every irreducible FH-submodule of V is finite-dimensional (see [5] Prop. 3). It follows that $\dim_F(\oplus_J V_j)$ is finite. But then $\oplus_J V_j$ lies in a direct sum of a finite number of irreducible FG-submodules of V. Consequently JG is a finite subset of I. But $w^G \subseteq \oplus_{j\in JG} W_j$ and the latter is finite dimensional. Therefore $\dim_F(Fw^G)$ is finite.◊

16. *In the situation of* **3.**, *suppose F is perfect and G is nilpotent and completely reducible. Then W is completely reducible as FG-module.*
Proof. By definition $V = V_0 \oplus V'$ and $W = V_0 \oplus W'$, see the proof of **3**. By **9.** and [5] Prop. 3 every cyclic subgroup of G is completely reducible, so by **14.** every element of G acts completely reducibly on W'. Consequently G does too by **9.** and **15.** Clearly G acts completely reducibly on V_0. Therefore W is a completely reducible FG-module.◊

17. *In the situation of* **3.**, *suppose $\mathrm{char} F = 0$ and G is soluble and completely reducible. Then W is a completely reducible FG-module.*
Proof. Repeat the proof of **16.**◊

18. *Let K and L be soluble, completely reducible, normal subgroups of the subgroup M of $GL(n, F)$. Under any one of the following three conditions, KL is completely reducible.*

a) char $F = 0$.

b) char $F = p > 0$, *F is perfect and $(K : K^0)$ and $(L : L^0)$ are prime to p.*

c) char $F = p > 0$, *F is perfect and K and L are locally nilpotent.*

(K^0 denotes the connected component of K containing the identity in the Zariski topology.)
Proof. In all three cases F is perfect, so ([1] p. 64 or [7] 1.24) we may assume that F is algebraically closed.

a) If K^0L^0 is completely reducible, then so is KL by Maschke's Theorem ([7] 1.5). Hence assume K and L are connected. Then K and L are diagonalizable and KL is triangularizable by the Lie–Kolchin Theoren ([7] 5.8). Hence $[K, L] \leq K \cap u(KL) = \langle 1 \rangle$ and KL is abelian. Therefore KL is completely reducible by [7] 7.1.

b) Here $(K : K^0)$ and $(L : L^0)$ are prime to p and $(KL : K^0L^0)$ divides the product of $(K : K^0)$ and $(L : L^0)$. Thus Maschke's Theorem still applies and we can repeat the proof of Part a).

c) By [7] 7.6 and 7.7 the indices $(K : K^0)$ and $(L : L^0)$ are coprime to p. Now apply Part b).◊

19. *Let K and L be soluble, completely reducible, normal subgroups of the subgroup M of $FGL(V)$. Suppose that either* char $F = 0$, *or F is perfect and K and L are locally nilpotent. Then KL is completely reducible.*
Proof. Let X and Y be finite subsets of K and L respectively and set $M_1 = \langle X, Y \rangle$, $K_1 = K \cap M_1$ and $L_1 = L \cap M_1$. Then K_1 and L_1 are

normal in M_1 and $M_1 = K_1L_1$. By **9.** and [5] Prop. 3 both K_1 and L_1 are completely reducible. Further, since M_1 is finitely generated and finitary, so $V = U \oplus C$ as FM_1-module, where $\dim_F U$ is finite and $C \leq C_V(M_1)$. By **18.** applied to the action of M_1 on U it follows that U and V are completely reducible as FM_1-modules.

By [4] or [9] the FKL-module $[V, KL]$ is completely reducible. Since K is completely reducible, V splits over $[V, KL]$ as FK-module. Hence

$$V = [V, KL] + C_V(K) = T \oplus C_V(K)$$

for some FKL-submodule T of $[V, KL]$; note that $C_V(K)$ is a KL-submodule of V. Since $T \leq [V, KL]$, so T is completely reducible as FKL-module, and since L is completely reducible, so $C_V(K)$ is completely reducible as FL-module and thus as FKL-module. Therefore V is completely reducible as FKL-module.◊

20. In **1.** it would be useful for induction purposes, if, when given that G is also completely reducible, one could deduce that W' is completely reducible as FG-module. Unfortunately this is false, which is the reason for the apparent detour through the previous series of lemmas. The simplest counterexample is probably the following. Suppose $\operatorname{char} F = 2$ and consider $G = GL(2,2) \leq GL(2, F)$. Then G is solube and absolutely irreducible. In fact G is dihedral of order 6, so let H be the normal subgroup of G of order 3. Then G/H acts faithfully and linearly on the centralizer space $C(\equiv W'$ here) of H in the matrix algebra $F^{2\times 2}$. Since $\operatorname{char} F = 2 = |G/H|$, so G/H acts unipotently, but not completely reducibly, on C.

21. *Suppose $\operatorname{char} F = 0$ and H is soluble of derived length d. Then there is a soluble normal subgroup $K \geq H$ of G of derived length d such that G/K is isomorphic to a finitary linear group over F.*

Proof. We induct on d; if $d = 0$, the claim is trivial. Let H_1 by the last non-trivial term (the $(d-1)$st term) of the derived series of H and set $K_1 = C_G(\operatorname{End}_{FH_1}V)$. Then $H_1 \leq K_1 \leq G$. Also H_1 is abelian, so $H_1 \subseteq \operatorname{End}_{FH_1}V$ and K_1 lies in the centre of $\operatorname{End}_{FH_1}V$. In particular K_1 is an abelian normal subgroup of G.

By Clifford's Theorem (applied to H) the group H_1 is completely reducible. Set $V = U_0 \oplus (\oplus_{j\in J} U_j)$, where $U_0 = C_V(H_1)$ and the U_j are non-zero homogeneous components of V as FH_1-module. Since H_1 is abelian and finitary, $\dim_F U_j$ is finite for each $j \in J$. Also $\operatorname{End}_{FH_1}V =$ $= (\operatorname{End}_F U_0) \oplus \prod_J \operatorname{End}_{FH_1}(U_j)$, where $\operatorname{End}_{FH_1}(U_j)$ is a matrix ring of finite degree over $D_j = \operatorname{End}_{FH_1}(I_j)$ for I_j any irreducible FH_1-submodule of U_j and the centralizer of $\text{End}_{FH_1}V$ in E is $F1_0 \oplus \prod_J \zeta_1(D_j)1_j$, where 1_0 is the identity on U_0, the element 1_j is the identity on U_j and $\zeta_1(D_j)$ denotes the centre of D_j. Consequently

$$C_{FGL(V)}(\text{End}_{FH_1}V) = \begin{cases} 1_0 + (\oplus_J \zeta_1(D_j^*)1_j) & \text{if } \dim_F U_0 \text{ is infinite} \\ F^*1_0 + (\oplus_J \zeta_1(D_j^*)1_j) & \text{if } \dim_F U_0 \text{ is finite.} \end{cases}$$

Either way this subgroup of $FGL(V)$ is abelian and completely reducible and consequently, by Clifford's Theorem, K_1 is completely reducible. By **3.** we can represent G/K_1 finitarily and faithfully on some F-space W and we take W to be the space constructed in the proof above of **3.** with V, G and K_1 in the place of V, G and H.

Taking $H = H_1$; that is, assuming H is abelian, we have proved the following.

22. *Suppose H is abelian. Then there is an abelian, completely reducible, normal subgroup K of G containing H such that G/K is isomorphic to a finitary linear group over F.*

We now return to the **proof of 21.** By **19.** the subgroup HK_1 of G is completely reducible. By **17.** the $F(HK_1/K_1)$-module W is completely reducible. By induction on d there is a soluble normal subgroup K/K_1 of G/K_1 containing HK_1/K_1 of derived length $d-1$ such that G/K is isomorphic to a finitary linear group over F. Clearly K is soluble of derived length at most $1 + (d-1)$ and $H \leq K$, so the derived length of K is exactly d. This completes the proof of **21.**◊

23. Question. Does there exist K as in **21.** such that K is completely reducible as a subgroup of $G \leq FGL(V)$? (Point **22.** above says 'yes' to this if H is abelian.)

24. *Suppose F is perfect and H is nilpotent of class c. Then there is a nilpotent normal subgroup $K \geq H$ of G of class c such that G/K is isomorphic to a finitary linear group over F.*

Proof. By **22.** we may assume that $c > 1$. We induct on c and copy the proof of **21.** with H_1 the centre of H. Thus set $K_1 = C_G(\text{End}_{FH_1}V)$. Here $H \leq \text{End}_{FH_1}V$ and K_1 is an abelian normal subgroup of G centralizing H. As in the proof of **21.** we have that HK_1 is completely reducible. By induction, cf. the proof of **21.**, there is a normal subgroup $K_2 \geq HK_1$ of G such that G/K_2 is isomorphic to a finitary linear group over F and K_2/K_1 is nilpotent of class $c-1$.

As in the proof of **21.**, the group K_1 is completely reducible. Let

$$V = U_0 \oplus (\oplus_{j\in J} U_j),$$

where here $U_0 = C_V(K_1)$ and where the U_j are non-zero homogeneous components of V as FK_1-module. Now U_j is finite F-dimensional and a direct sum of a finite number, r_j say, of copies of an irreducible FK_1-module I_j. Set $D_j = \mathrm{End}_{FK_1} I_j$. Then

$$C = FGL(V) \cap \mathrm{End}_{FK_1} V = FGL(U_0) \times (\times_J S_j),$$

where S_j is isomorphic to $GL(r_j, D_j)$, acting in the obvious way on U_j. Thus U_0 and each U_j are irreducible as FC-modules and hence C is completely reducible. By **3.** the group $N_{FGL(V)}(C)/C$ is isomorphic to some finitary linear group over F. Then so too is $G/C_G(K_1)$ ($\cong GC/C \leq$ $\leq N_{FGL(V)}(C)/C$).

Set $K = K_2 \cap C_G(K_1)$. Since K_2/K_1 is nilpotent of class $c - 1$, so K is nilpotent of class at most c. But H centralizes K_1, so $H \leq K$ and K is nilpotent of class exactly c. Finally if X and Y are F-spaces such that G/K_2 is embeddable in $FGL(X)$ and $G/C_G(K_1)$ is embeddable in $FGL(Y)$ then G/K is embeddable into $FGL(X \oplus Y)$. The proof is complete.◊

Note that the Theorem follows from **21.**, **22.** and **24.**

We have yet to exhibit any example of a Zariski-closed normal subgroup $H \leq G \leq FGL(V)$ such that G/H is not isomorphic to any finitary linear group. First we summarize the basic properties of the Zariski topology.

25. The Zariski Topology. With V, as above, a vector space over the field F, set, for the moment, $G = FGL(V)$. Suppose $V = U \oplus W$ as F-space, where $\dim_F U$ is finite. Set $G(U, W) = GL(U) + 1_W \leq G$. Now $GL(U)$ carries its usual Zariski topology. Transfer this to $G(U, W)$ via the obvious isomorphism of $GL(U)$ to $G(U, W)$. Call a subset X of G *(Zariski) closed* in G if and only if $X \cap G(U, W)$ is closed in $G(U, W)$ for all decompositions $V = U \oplus W$ with $\dim_F U$ finite. This defines a topology on G that coincides with the usual Zariski topology on G whenever $\dim_F V$ is finite. Of course we call this topology the *Zariski topology* on G.

This topology does not, in general, make G into a CZ-group (see[7] Chap. 5), but we do at least have the following.

a) Each finite subset of G is closed in G.

b) If $a \in G$, then the following four maps of G into itself are continuous: $x \mapsto ax$, $x \mapsto xa$, $x \mapsto x^{-1}$, $x \mapsto x^{-1}ax$.

c) If X is a subset of G, then $C_G(X)$ is a closed subgroup of G.

d) If X is a closed subset of G, then $N_G(X)$ is a closed subgroup of G.

e) Let X be a subspace of V. Then $N_G(X)$ and $C_G(X)$ are closed in G.

f) If X is a subgroup of G with closure $X^\frown$ in G, then $X^\frown$ is a subgroup of G and if X is normal in G, respectively soluble of derived length d, respectively nilpotent of class c, then so is $X^\frown$.

26. Some examples. Returning to the notation $H \triangleleft G \leq FGL(V)$ of the introduction, if it was true, that G/H is isomorphic to a finitary linear group over F whenever H is Zariski-closed in G (or at least when H is also completely reducible), then the Theorem and the Proposition would follow almost at once. Unfortunately, the example, which we now construct, shows this is not the case.

For $i = 1, 2, \dots$ let

$$G_i =< x_i, y_i \mid x_i^{p^2} = y_i^{p^2} = 1 \text{ and } G \text{ nilpotent of class } 2 > .$$

Here p is any prime. Let q be a characteristic and F any field of characteristic q. Clearly G_i acts faithfully on a vector space V_i over F of degree p^6. Let $V = \oplus_{i\geq 1} V_i$ and $G = \times_{i\geq 1} G_i$, regarded as a subgroup of $FGL(V)$ in the obvious way. Let $z_i = [x_i, y_i]$ for each i; so $< z_i >$ is the centre of G_i and has order p^2. (To see this consider the unitriangular group $\mathrm{Tr}_1(3, \mathbb{Z}/p^2)$.) Set

$$H =< z_i^{-1} z_j : i, j \geq 1 >= \times_{i>1} < z_{i-1}^{-1} z_i > .$$

Let $V = U \oplus W$ as F-space, where $\dim_F U$ is finite. Then $U \leq$ $\leq \oplus_{j\in J} V_j$ for some finite subset J of I. Let $G_0 = G \cap G(U, W)$. Since $[V, G_0] \leq U$, so G_0 centralizes $\oplus_{i\notin J} V_i$. Thus $G_0 \leq C_G(\oplus_{i\notin J} V_i) = \times_{j\in J} G_i$, which is finite. Therefore $H \cap G(U, W)$ is finite and hence closed in $G(U, W)$. Consequently H is Zariski-closed in $FGL(V)$.

If $q = p$, then G is unipotent. If $q \neq p$, then V_i is comletely reducible as FG_i-module, so G and hence H (by Clifford's Theorem) are completely reducible subgroups of $FGL(V)$. Also G and G/H are nilpotent of class 2 and p-groups of finite expotent (p^2 if $p > 2$ and 8 if $p = 2$).

By 3.6 of [2] the group G/H is not isomorphic to any finitary linear group of characteristic p. Suppose X is a finitary linear group of characteristic not p that is isomorphic to G/H. Then X has trivial unipotent radical, so we may choose X completely reducible. Since X is nilpotent and finitary, each irreducible constituent of X is finite dimensional ([5]

Prop. 3). Now a nilpotent irreducible linear group of finite degree and finite expotent is finite. Thus X is residually finite. The centre Z of X has order p^2. Thus there is a normal subgroup Y of X of finite index with $Y \cap Z = < 1 >$. It follows from the nilpotence of X that $Y = < 1 >$. But X/Z is infinite, being a direct product of infinitely many cyclic groups of order p^2. Therefore G/H is not isomorhic to any finitary linear group. Choosing $q \neq p$ gives Ex. a) and choosing $q = p$ gives Ex. b).

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