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ON CONVOLUTION TYPE FUNCTIONAL EQUATIONS

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Abstract: In this paper we deal with relations between systems of convolution type functional equations on discrete abelian groups. If a set Λ of finitely supported complex measures on the abelian group G is given, then the system of equations $f * \lambda = 0$ (for all λ in Λ) is called the system of convolution type functional equations (associated with Λ). We are looking for complex valued solutions of this system. In the applications the following problem arises: if two systems of convolution type functional equations are given, how to decide if one of them implies the other, or how to decide if they are equivalent? As the solution spaces of systems of convolution type functional equations are translation invariant linear spaces, closed with respect to the topology of pointwise convergence, hence the methods of spectral synthesis can be applied. As spectral synthesis holds for finitely generated abelian groups, in order to apply these methods, the above problem should be reduced to finitely generated subgroups. This paper exhibits the possibility of this reduction. It turns out that the equivalence of two systems depends on the relation between their spectral sets. In addition we give a necessary and sufficient condition for the validity of spectral synthesis on discrete abelian groups.

In the paper $\mathbb{C}$ denotes the set of complex numbers. Let G be an abelian group. Homomorphisms of G into the additive group of complex

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numbers are called *additive functions* and homomorphisms of G into the multiplicative group of nonzero complex numbers are called *exponential functions* or simply *exponentials.* Products of additive functions and exponentials are called *exponential monomials.* As a product of exponentials is an exponential too, the general form of exponential monomials is

$$x \mapsto a_1^{\alpha_1}(x)a_2^{\alpha_2}(x)\dots a_n^{\alpha_n}(x)m(x),$$

where $a_1, a_2, \dots, a_n : G \to \mathbb{C}$ are additive functions, $m : G \to \mathbb{C}$ is an exponential and $n, \alpha_1, \alpha_2, \dots, \alpha_n$ are non-negative integers. If here m is identically 1, then we call the above function a *monomial.* Linear combinations of monomials are called *polynomials* and linear combinations of exponential monomials are called *exponential polynomials.* Hence exponential polynomials are elements of the algebra generated by all additive and exponential functions.

If G is an abelian group then $\mathcal{C}(G)$ denotes the set of all complex valued functions on G. With the pointwise operations $\mathcal{C}(G)$ is a linear space and equipped with the topology of pointwise convergence it is a locally convex topological vectorspace. The dual of $\mathcal{C}(G)$ can be identified with the space $\mathcal{M}_c(G)$ of all finitely supported complex measures.

A closed translation invariant subspace of $\mathcal{C}(G)$ is called a *variety.* The subspaces $\{0\}$ and $\mathcal{C}(G)$ are *trivial varieties.* For any f in $\mathcal{C}(G)$ the closed subspace spanned by all translates of f is evidently a variety, which is called the *variety generated by* f. Another important example for variety is $V(\lambda)$, the set of all functions f in $\mathcal{C}(G)$, satisfying the equation $f * \lambda = 0$, where λ is in $\mathcal{M}_c(G)$. Obviously, the intersection of any family of varieties is a variety, too. Hence, if Λ is a set of finitely supported measures, then the set of all functions f in $\mathcal{C}(G)$ satisfying $f * \lambda = 0$ for all λ in Λ is a variety, which we denote by $V(\Lambda)$. We define $V(\emptyset) = \mathcal{C}(G)$. The system of equations $f * \lambda = 0$ for all λ in Λ is the *system of convolution type functional equations associated with* Λ. In this interpretation $V(\Lambda)$ is the *solution space* of this system. Hence the solution space of any system of convolution type functional equations is a variety. Conversely, it follows from Lemma 8.1. in [3] that any variety is the solution space of some system of convolution type functional equations. In other words, for every variety V there exists a set Λ of finitely supported complex measures such that $V = V(\Lambda)$. Indeed, for Λ we can take the *annihilator* of V in $\mathcal{M}_c(G)$, that is the set of all finitely supported complex measures, which vanish on the variety V.

The basic problem concerning varieties is the following: how far the exponential monomials belonging to a variety determine this variety?

The set of exponentials contained in a variety is called the *spectrum of the variety.* The set of all exponential monomials in a given variety is the *spectral set of the variety.* The spectral set of the variety generated by an f in $\mathcal{C}(G)$ is called the *spectral set of* f. If Λ is a set of finitely supported complex measures on g, then the spectral set of $V(\Lambda)$ is called the *spectral set of* Λ. The basic problem for varieties can be formulated as follows: is the linear hull of the spectral set of a variety dense in the variety? If so, then we say that *spectral synthesis holds for the variety.* If spectral synthesis holds for every nontrivial variety in $\mathcal{C}(G)$, then we say that *spectral synthesis holds for* G. In terms of systems of convolution type functional equations spectral synthesis means, that if a set Λ of finitely supported complex measures on G is given, then any f satisfying

$$f * \lambda = 0 \text{ for all } \lambda \text{ in } \Lambda$$

is the pointwise limit of linear combinations of exponential monomial solutions of this system. A basic result concerning spectral synthesis is the following.

Theorem 1 [2, Lefranc]. *If G is a finitely generated abelian group then spectral synthesis holds for G.*

In [1] there is a similar theorem for any abelian group. Unfortunately, there is a gap in the proof. The problem concerning spectral synthesis for arbitrary abelian groups is still open. However, it turns out that in most cases the problem of application of spectral synthesis for systems of convolution type functional equations on abelian groups can be reduced to finitely generated subgroups.

Let Λ and Γ be two sets of finitely supported complex measures on G. We say that Λ *implies* Γ if $V(\Lambda)$ is a subset of $V(\Gamma)$. In other words, Λ implies Γ if and only if any solution of the system of convolution type functional equations associated with Λ is a solution of the system of convolution type functional equations associated with Γ, too. We say that Λ and Γ are *equivalent*, if they mutually imply each other, that is, if $V(\Lambda) = V(\Gamma)$. This means, that Λ and Γ are equivalent if and only if the systems of convolution type functional equations associated with Λ and with Γ have the same solutions.

Let G be an abelian group and let $F \subseteq G$ be a subgroup. Let Λ be a set of finitely supported complex measures on G. We define the *restriction of* Λ *to* F as the set of all measures λ in Λ, for which the support of λ is contained in F. It is denoted by Λ_F.

Theorem 2. *Let G be an abelian group and let Λ, Γ be sets of finitely supported complex measures on G. If the restriction of Λ to any finitely* *generated subgroup of G implies the restriction of Γ to the same subgroup, then Λ implies Γ.*

Proof. Assuming the condition of the theorem suppose that Λ does not imply Γ. This means that $V(\Lambda)$ is not contained in $V(\Gamma)$, hence there exists an element f in $V(\Lambda)$ which does not belong to $V(\Gamma)$. In other words we have

$$f * \lambda = 0$$

for all λ in Λ and there exists a γ_0 in Γ for which

$$f * \gamma_0 \neq 0.$$

Thus there exists an x_0 in G for which

$$\big(f * \gamma_0\big)(x_0) \neq 0.$$

Let F be the subgroup generated by x_0 and the support of γ_0. Then F is a finitely generated subgroup of G. Obviously, the restriction $f|_F$ of f to F belongs to $V(\Lambda_F)$. On the other hand we have

$$\big(f|_F * \gamma_0\big)(x_0) = \big(f * \gamma_0\big)(x_0) \neq 0,$$

and γ_0 belongs to Γ_F, hence $f|_F$ does not belong to $V(\Gamma_F)$, that is, Λ_F does not imply Γ_F. This is a contradiction and hence the theorem is proved. ◇

Corollary 3. *Let G be an abelian group and let Λ, Γ be sets of finitely supported complex measures on G. Then Λ and Γ are equivalent if their restrictions to any finitely generated subgroup of G are equivalent.*

Theorem 4. *Let G be a finitely generated abelian group and let Λ and Γ be sets of finitely supported complex measures on G. Then Λ implies Γ if and only if the spectral set of Λ is a subset of the spectral set of Γ.*

Proof. The necessity of the condition is obvious. For the sufficiency we suppose that any exponential monomial in $V(\Lambda)$ belongs to $V(\Gamma)$. Then the variety V generated by all exponential monomials in $V(\Lambda)$ is contained in $V(\Gamma)$. Using Th. 1 on spectral synthesis for finitely generated abelian groups, V is equal to $V(\Lambda)$, and the theorem is proved. ◇

Corollary 5. *Let G be a finitely generated abelian group and let Λ and Γ be sets of finitely supported complex measures on G. Then Λ and Γ are equivalent if the spectral sets of Λ and Γ are the same.*

Th. 4 has the following obvious generalization.

Theorem 6. *Let G be an abelian group and suppose that spectral synthesis holds for G. Let Λ and Γ be sets of finitely supported complex measures on G. Then Λ implies Γ if and only if the spectral set of Λ is a subset of the spectral set of Γ.*

The proof of this theorem is identical with that of Th. 4. Obviously, the same kind of generalization can be formulated with respect to Cor. 5.

Corollary 7. *Let G be an abelian group and suppose that spectral synthesis holds for G. Let Λ and Γ be sets of finitely supported complex measures on G. Then Λ and Γ are equivalent if and only if the spectral sets of Λ and Γ are the same.*

The following is the converse of the statement of Th. 4.

Theorem 8. *Let G be an abelian group and suppose that for any sets Λ, Γ of finitely supported complex measures on G if the spectral set of Λ is a subset of the spectral set of Γ, then Λ implies Γ. Then spectral synthesis holds for G.*

Proof. Suppose that spectral synthesis does not hold for G. This means that there exists a nontrivial variety V in $\mathcal{C}(G)$ such that the closed linear hull of all exponential monomials in V is a proper subvariety V_0 of V. Then the spectral set of V is a subset of the spectral set of V_0 (in fact, they are equal), hence by assumption, V is contained in V_0. It follows that $V_0 = V$, which contradicts to the fact, that V_0 is a proper subvariety. $\Diamond$

References

[1] ELLIOT, T.S.: Two notes on spectral synthesis for discrete Abelian groups, *Proc. Cambridge Phil. Soc.* **61** (1965), 617–620.

[2] LEFRANC, M.: L'analyse harmonique dans $\mathbb{Z}^n$, *C.R. Acad. Sci. Paris* **246** (1958), 1951–1953.

[3] SZÉKELYHIDI, L.: Convolution Type Functional Equations on Topological Abelian Groups, World Scientific, Singapore, New Jersey, London, Hong Kong, 1991.