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NEAR-RINGS IN WHICH EACH PRIME FACTOR IS SIMPLE

Gary F. **Birkenmeier**

Department of Mathematics, University of Southwestern Louisiana, P. O. Box 41010, Lafayette, LA 70504-1010, USA.

Nico J. **Groenewald**

Department of Mathematics, University van Port Elizabeth, P. O. Box 1600, Port Elizabeth, South Africa.

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Abstract: In this paper we investigate connections between the condition that every prime ideal is maximal and various generalizations of von Neumann regularity. As a corollary of our results we show that if N is a reduced zero-symmetric right near-ring, then every prime ideal is maximal if and only if N is left weakly regular (i.e., $x \in \langle x \rangle x$, for all $x \in N$, where $\langle x \rangle$ denotes the ideal generated by x).

Throughout this paper all near-rings are zero symmetric right near-rings, and N denotes such a near-ring. It can be shown that if $N^2 = N$, then every maximal ideal is a prime ideal. Since, in general, prime ideals are not maximal even in near-rings with unity, it is natural to ask: when is every prime ideal of N also a maximal ideal of N? Near-rings satisfying this condition (equivalently, every prime factor is simple) are said to satisfy the *pm condition.* Surprisingly, the *pm* condition (a condition on ideals) has been shown to be equivalent to various generalizations of von Neumann regularity (a condition on elements) for large classes of rings [2], [4], [8], [9], [11], [12], [13], [14], [19], [20], [28], [29], [31] and [32]. The survey paper [5] gives an overview of the research in this area.

For the case of commutative rings, the first clearly established equivalence between the *pm* condition and a generalization of von Neumann regularity seems to have been made by Storrer [28] in the following result. If R is a commutative ring with unity, then the following statements are equivalent: (1) R is π-regular; (2) $R/\mathcal{P}(R)$ is von Neumann regular ($\mathcal{P}(R)$ is the prime radical of R); (3) R satisfies the *pm* condition. For a near-ring N with unity it is well known [24, p.349] that if N is strongly regular then N satisfies the *pm* condition. In the paper we will investigate the connections between the *pm* condition and various generalizations of von Neumann regularity in the class of near-rings. In particular, we will extend the main results of [8] to near-rings. We provide examples which illustrate the contrast between the ring and the near-ring cases for our results.

Let $\mathcal{P}_0(N)$ denote the prime radical and $\mathcal{N}(N)$ the set of nilpotent elements of the near-ring N. From [6], an ideal I of N is a 2-primal ideal of N if $\mathcal{P}_0(N/I) = \mathcal{N}(N/I)$. If I is the zero ideal of N, then N is a 2-primal near-ring. (This is equivalent to $\mathcal{P}_0(N) = \mathcal{N}(N)$. A near ring is said to be *reduced* if $\mathcal{N}(R) = 0$. Recall from [24] that an ideal P is called a *minimal prime ideal of an ideal* I if P is minimal in the set of all prime ideals containing I. If I is the zero ideal, then P is called a *minimal prime ideal* of N. By $\mathcal{B}(I)$ we denote the intersection of all prime ideals of N containing I. From [16], $\mathcal{B}(I)$ is the intersection of all minimal prime ideals in N containing I. An ideal I of N, denoted by $I \lhd N$, is a *completely prime ideal (completely semi-prime ideal)* if for $a, b \in N$, $ab \in I$ implies $a \in I$ or $b \in I$ ($a^2 \in I$ implies $a \in I$). The completely prime radical $\mathcal{P}_c(N)$ of the near-ring N is the intersection of all the completely prime ideals of N. If follows from [17] that $\mathcal{P}_c(N)$ is completely semi-prime ideal of N. Moreover from [7], N is 2-primal if and only if $\mathcal{P}_0(N) = \mathcal{P}_c(N)$.

We use $\mathcal{N}_r(N)$, $\mathcal{J}_2(N)$ and $\mathcal{G}(N)$ to represent the nilradical of N, $\mathcal{J}_2$-radical of N and the Brown-McCoy radical of N, respectively. N is said to fulfill the *insertion-of-factors property* (IFP) provided that for all $a, b, x \in N$, then $ab = 0$ implies $axb = 0$. Also for $X \subseteq N$, $(0 : X)$ and $\langle X \rangle$ denote the left annihilator of X and the ideal of N generated by X, respectively. For other notation and/or terminology see [24].

1. Preliminaries

In this section we discuss the various generalizations of von Neumann regularity which will be used in our main results in section 3.

In [25], Ramamurthi defined weakly regular rings, and in [18] Gupta defined weakly π-regular rings. Jat and Choudhary [21] extended weak regularity to near-rings, and Goyal and Choudhary [15] did likewise for π-regularity. Furthermore Ramakotaiah [26] gave a nonring example of a π-regular subnear-ring of $M_0(\mathbb{Z}_4)$. The following definitions will be used in this paper (note that we are introducing the concept of a left pseudo π-regular near-ring).

Definition 1.1. (i) N is a said to be *left (right) weakly regular* if $x \in \langle x \rangle x$ ($x \in x \langle x \rangle$) for all $x \in N$.

(ii) N is said to be *π-regular* if for every $x \in N$ there exists a natural number $n = n(x)$ such that $x^n \in x^n N x^n$.

(iii) N is said to be *left (right) weakly π-regular* if for every $x \in N$ there exists a natural number $n = n(x)$ such that $x^n \in \langle x^n \rangle x^n$ ($x^n \in x^n \langle x^n \rangle$).

(iv) We say N is *left (right) pseudo π-regular* if for every $x \in N$ there exists a natural number $n = n(x)$ such that $x^n \in \langle x \rangle x^n$ ($x^n \in x^n \langle x \rangle$).

In the above definitions if N satisfies both the left and right version, then the adjective "left" or "right" is omitted. Observe that we have the following implications from Definition 1.1: (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Moreover the classes of near-rings of Definition 1.1 as well as the class of *pm* near-rings are closed under homomorphic images. Observe that if N is left (right) weakly regular and I is an ideal of N, then $I = I^2$. From [23], N is left strongly regular if for all $x \in N$, there exists $a \in N$ with $x = ax^2$. Hence if N is left strongly regular, then N is left weakly regular. Furthermore, from [21], N is bipotent if $Na = Na^2$ for $a \in N$. So if N is bipotent and $a \in N$, then $\langle a \rangle a^2 \subseteq Na^2 = (Na)a = (Na^2)a = (Na)a^2 \subseteq \langle a \rangle a^2$. Thus $Na = \langle a \rangle a^2$, so $a^2 \in \langle a \rangle a^2$ for every $a \in N$. Hence every bipotent near-ring is left pseudo π-regular.

Lemma 1.2. (i) *Let N be reduced. Then N is left weakly regular if and only if N is left pseudo π-regular.*

(ii) *Let N be commutative. Then N is weakly π-regular if and only if N is pseudo π-regular.*

Proof. (i) Clearly if N is left weakly regular then N is left pseudo π-regular. So assume N is left pseudo π-regular. Let $a \in N$. Then there exists $s \in \langle a \rangle$ and a natural number n such that $a^n = sa^n$. If $n = 1$, we are finished. For $n > 1$, then $(a - sa)a^{n-1} = 0$. Since 0 is a completely semiprime ideal, [17, Lemma 2.1] yields $(a - sa)a = 0 = a(a - sa)$. Hence $(a - sa)^2 = 0$. So $a = sa \in \langle a \rangle_a$. Therefore N is left weakly regular.

(ii) Assume N is left pseudo π-regular. Let $a \in N$. There exists $s \in \langle a \rangle$ and a natural number n such that $a^n = sa^n = s(sa^n) = s^2a^n = = s^na^n \in \langle a^n \rangle a^n$. Hence N is left weakly π-regular. The converse is clear. ◊

The following result generalizes [15, Th. 1.14].

Proposition 1.3. *Let $a \in N$.*

(i) *$a^k \in Na^{k+1}$ for some positive integer k if and only if the descending chain $Na \supseteq Na^2$... stabilizes after a finite number of steps.*

(ii) *If N is finite, then N is left and right weakly π-regular.*

Proof. (i) Assume $a^k \in Na^{k+1}$. Then there exists $x \in N$ such that $a^k = xa^{k+1}$. Then $Na^k = Nxa^{k+1} \subseteq Na^{k+1} \subseteq Na^k$. Hence the chain stabilizes. Conversely assume $Na^m = Na^{m+1}$. Then there exists $y \in N$ such that $a^{m+1} = aa^m = ya^{m+1}$. There exists $y_1 \in N$ such that $ya^m = = y_1a^{m+1}$. So $a^{m+1} = (ya^m)a = (y_1a^{m+1})a \in Na^{m+2}$. Take $k = m + 1$.

(ii) By part (i), there exists a positive integer k and $x \in N$ such that $a^k = xa^{k+1} = xa^ka = x(xa^{k+1})a = x^2a^ka^2 = ... = x^ka^ka^k \in \langle a^k \rangle a^k$. Hence N is left weakly π-regular. Since part (i) is left-right symmetric, N is also right weakly π-regular. ◊

Observe that every finite ring satisfies the *pm* condition. However there are finite *d.g.* prime near-rings with unity which are not simple [22]. Thus determining when a finite near-ring satisfies the *pm* condition is a notrival problem. Also from [3] there exist uncountable *d.g.* near-rings with unity, but with only finitely many N-subgroups. By Prop. 1.3(ii) such near-rings are left weakly π-regular. Also note that integral near-rings which are finite or simple illustrate Lemma 1.2(i), and near-rings of nilpotent index two illustrate Lemma 1.2(ii).

Proposition 1.4. *Let N be a near-ring with left unity e, and k and n are natural numbers.*

(i) *If $N = (0 : a^n) + \langle a^k \rangle$, then $a^n \in \langle a^k \rangle a^n$.*

(ii) *If $(0 : a^n) \triangleleft N$ and $a^n \in \langle a^k \rangle a^n$, then $N = (0 : a^n) + \langle a^k \rangle$.*

Proof. (i) There exists $v \in (0 : a^n)$ and $s \in \langle a^k \rangle$ such that $e = v + s$. Then $a^n = va^n + sa^n = sa^n \in \langle a^k \rangle a^n$.

(ii) There exists $s \in \langle a^k \rangle$ such that $a^n = sa^n$. So $(e - s) \in (0 : a^n)$. Then for any $t \in N$, we have $t = (e - s + s)t = (e - s)t + st$. Since $(0 : a^n) \triangleleft N$, $(e - s)t \in (0 : a^n)$. Therefore $N = (0 : a^n) + \langle a^k \rangle$. ◊

Corollary 1.5. *Let N be an IFP near-ring with a left unity. Then N is left pseudo (weakly) π-regular if and only if for every $a \in N$ there exists a natural number $n = n(a)$ such that*

$$N = (0 : a^n) + \langle a \rangle, \quad (N = (0 : a^n) + \langle a^n \rangle).$$

Proof. Since N is IFP, then $(0 : x) \triangleleft N$ for every $x \in N$. Now the result is an immediate consequence of Prop. 1.4.◊

Proposition 1.6. *Let I be any proper ideal of left pseudo π-regular near-ring N. Every nonzero element of I is a divisor of zero.*

Proof. Let $0 \neq a$ be any element of the ideal I. Assume a is not a divisor of zero. Since N is left π-weakly regular there exists $n(a)$, a positive integer, such that $a^n \in \langle a \rangle a^n$. Hence $a^n = xa^n$ for some $x \in \langle a \rangle$. For every $z \in N$ we have $za^n = zxa^n$. Hence $(z - zx)a^n = 0$. Since a is not a divisor of zero, we have $z = zx \in I$. Hence $N = I$ which is a contradiction.◊

Corollary 1.7. *If N is weakly π-regular with nonzero divisors of zero, then N is simple.*

2. Completely prime ideals and the *pm* condition

Proposition 2.1. *$\rho(N)$ is completely semiprime if and only if every prime ideal which is minimal among the prime ideals containing $\rho(N)$ is completely prime (where $\rho(N) = \mathcal{P}_0(N), \mathcal{N}_r(N), \mathcal{J}_2(N)$ or $\mathcal{G}(N)$).*

Proof. Let P be a prime ideal which is minimal amongst the prime ideals containing $\rho(N)$. Now clearly, $P/\rho(N)$ is a minimal prime ideal of $\bar{N} = = N/\rho(N)$. Since $\rho(N)$ is completely semiprime, $\bar{N}$ is reduced. Since $\bar{N}$ is reduced, it is also 2-primal and from [7, Cor. 1.3] we have that $P/\rho(N)$ is a completely prime ideal of $\bar{N}$. Hence $N/P \cong (N/\rho(N))/(P/\rho(N))$ is a completely prime near-ring and consequently P is a completely prime ideal of N.

Now let B be the intersection of all the prime ideals of N which are minimal among prime ideals of N containing $\rho(N)$. Let D be the intersection of all the prime ideals of N containing $\rho(N)$. For $\rho(N) = \mathcal{P}_0(N)$ we have $\mathcal{P}_0(N) = B$ and from our assumption $\mathcal{P}_0(N)$ is the intersection of completely prime ideals and hence $\mathcal{P}_0(N)$ is completely semiprime.

Case $\rho(N) = \mathcal{N}_r(N)$. Recall from [30] that $\mathcal{N}_r(N)$ is the intersection of all s-prime ideals and that each s-prime ideal is also a prime ideal of N which contains $\mathcal{N}_r(N)$. Hence $\mathcal{N}_r(N) \subseteq B = D \subseteq \subseteq \cap \{s\text{-prime ideals of } N\} = \mathcal{N}_r(N)$.

Case $\rho(N) = \mathcal{J}_2(R)$. Recall that a 2-primitive ideal of N is a prime ideal of N which contains $\mathcal{J}_2(N)$. Now $\mathcal{J}_2(N) \subseteq B = D \subseteq \subseteq \cap \{s\text{-primitive ideals of } N\} = \mathcal{J}_2(N)$.

Case $\rho(N) = \mathcal{G}(N)$. From [1] we know that $\mathcal{G}(N) = \cap\{M \triangleleft N : N/M \text{ is a simple near-ring with identity}\}$. Each of the ideal in the intersection is clearly a prime ideal and contains $\mathcal{G}(N)$. Hence $\mathcal{G}(N) \subseteq B = = D \subseteq \cap\{M : N/M$ is simple with identity$\} = \mathcal{G}(N)$.

Thus in all four cases $B = \rho(N)$ and since B is completely semiprime, then so is $\rho(N)$.◊

The following well known result is an immediate corollary of Prop. 2.1.

Corollary 2.2. *Let N be a reduced near-ring. Every minimal prime ideal of N is completely prime.*

Proof. Since N is reduced, $\mathcal{P}_0(N) = 0$. The corollary now follows from Prop. 2.1.◊

Proposition 2.3. *If $\rho(N)$ is a completely semiprime ideal of N and $N/\rho(N)$ is left pseudo π-regular, then N/P is a simple integral near-ring with a right unity for every prime ideal P of N with $\rho(N) \subseteq P$ (where $\rho(N) = \mathcal{P}_0(N), \mathcal{N}_r(N), \mathcal{J}_2(N)$ or $\mathcal{G}(N)$).*

Proof. Let P be any prime ideal of N such that $\rho(N) \subseteq P$. Now there exists a prime ideal X of N which is minimal among prime ideals containing $\rho(N)$ and $X \subset P$. From Prop. 2.1, X is completely prime. Let $\bar{N} = N/X$.

Since X is a completely prime ideal, $\bar{N}$ is an integral near-ring. We show that N is simple with a right unity. Let $0 \neq I \triangleleft \bar{N}$ and $0 \neq v \in I$. Since $\bar{N}$ is weakly π-regular, there exists $y \in \langle v \rangle$ such that $v^k = yv^k$. Now we have $yv^k = y^2v^k$. Hence $(y - y^2)v^k = 0$. Since $\bar{N}$ is an integral near-ring and $v \neq 0$, we have $y = y^2$. Hence for any $t \in \bar{N}$ we have $ty = ty^2$. Now $(t - ty)y = 0$ and therefore $t = ty$. Hence y is a right unity of $\bar{N}$. Now also $t = ty \in t\left\langle v^k \right\rangle \subseteq tI \subseteq I$. Hence $\bar{N} = I$ and, therefore, $\bar{N}$ is simple with right unity y. Hence X is maximal and $X \subseteq P$, therefore $X = P$.◊

Corollary 2.4. *If the near-ring N is 2-primal and $N/\mathcal{P}_0(N)$ is left pseudo π-regular, then every prime ideal of N is maximal.*

Proposition 2.5. *If $I \triangleleft N$, then $\mathcal{B}(I)$ is completely semiprime if and only if every minimal prime ideal of I is completely prime. In particular, if N also satisfies the pm condition, then every prime ideal containing I is completely prime.*

Proof. The result is a consequence of [6, Lemma 2.2(v)] and [7, Th. 1.2].◊

Corollary 2.6. *If N is 2-primal (e.g., if N is reduced) and satisfies the pm condition, then every prime ideal is completely prime.*

3. Left weakly regular near-rings

In this section we characterise reduced left weakly regular near-rings.

Lemma 3.1. *If I is a completely semi-prime ideal of the near-ring N and $x_1, x_2, ..., x_n \in I$ then $x_{\sigma(1)}, ..., x_{\sigma(n)} \in I$, where σ is any permutation of $\{1, 2, ..., n\}$.*

Proof. This follows by applying Lemma 2.1 of [17].$\diamond$

Proposition 3.2. *Let N be an IFP right near-ring with left unity e such that every completely prime ideal is maximal. Let $a \in N$ such that $(0 : a)$ is a 2-primal ideal of N, there exists $s \in \langle a \rangle$ such that:*

(i) *$a^3 = sa^3 + x$ where $s \in \langle a \rangle$ and $x \in \mathcal{N}(N)$.*

(ii) *If $a^3(e - s)a^3 = (e - s)a^k$ for some k, then there exists m such that $a^m \in \langle a \rangle a^m$.*

Proof. (i) Let $0 \neq a \in N$. Since N has IFP, it follows from [24, p. 289] that $(0 : a) \triangleleft N$. Let $\bar{N} = N/(0 : a)$. It is easy to see that every completely prime ideal of $\bar{N}$ is also maximal. Let M be the multiplicative semigroup generated by all elements of the form $\bar{a} = \bar{x}\bar{a}$ where $x \in \langle a \rangle$.

We claim $\mathcal{P}_0(\bar{N}) \cap M \neq \varnothing$. To see this, assume $\mathcal{P}_0(\bar{N}) \cap M = \varnothing$. Since $\mathcal{P}_0(\bar{N}) = \mathcal{N}(\bar{N})$ (because $(0 : a)$ is a 2-primal ideal - see [7]), $\mathcal{P}_0(N)$ is a completely semiprime ideal of N. If follows from [17, Lemma 3.1] that there exists a completely prime ideal $\bar{P}$ and $\bar{N}$ such that $\bar{P} \cap M = \varnothing$. Now $\langle \bar{a} \rangle \subseteq \bar{P}$ or there exists $\bar{\alpha} \in \langle \bar{a} \rangle$ such that $\bar{\alpha} \notin \bar{P}$. If $\langle \bar{a} \rangle \subseteq \bar{P}$, then $\bar{a} - \bar{x}\bar{a} \in \bar{P} \cap M \neq \varnothing$, a contradiction . So assume there exists $\bar{\alpha} \in \langle \bar{a} \rangle$ such that $\bar{\alpha} \notin \bar{P}$. Since $\bar{P}$ is maximal, we have $\bar{P} + \langle \bar{a} \rangle = \bar{N}$. So $\bar{e} = \bar{p} + \bar{t}$, where $\bar{p} \in \bar{P}$ and $\bar{t} \in \langle \bar{\alpha} \rangle \subseteq \langle \bar{a} \rangle$. From this we have $\bar{a} - \bar{t}\bar{a} = (\bar{e} - \bar{t})\bar{a} = \bar{p}\bar{a} \in \bar{P} \cap M$, a contradiction. Hence $\mathcal{P}_0(\bar{N}) \cap M \neq \varnothing$. So,

$$(\bar{a} - \bar{t}_1\bar{a})(\bar{a} - \bar{t}_2\bar{a})...(\bar{a} - \bar{t}_n\bar{a}) \in \mathcal{P}_0(N)$$

for some $\bar{t}_i \in \langle \bar{a} \rangle$. By Lemma 3.1, there exists $\bar{s} \in \langle \bar{a} \rangle$ such that

$$(\bar{e} - \bar{s})\bar{a}^n \in \mathcal{P}_0(\bar{N}).$$

From [17, Lemma 2.1(iii)],

$$(\bar{e} - \bar{s})\bar{a} = \bar{k}\bar{a} \in \mathcal{P}_0(\bar{N}).$$

So $(\bar{e} - \bar{a})\bar{a}^2 = \bar{k}\bar{a} \in \mathcal{P}(\bar{N})$. Thus $(\bar{a} - \bar{s}\bar{a} - \bar{k})\bar{a} = \bar{0}$. Hence $(a - sa - k)aa = 0$, where $k^j a = 0$ for some positive integer j. Then $a^3 = sa^3 + ka^2$. Since N has IFP, $(ka)^j = 0$ and $(ka^2)^j = 0$. Therefore, $ka^2 \in \mathcal{N}(N)$.

(ii) By part (i) there exists n such that $0 = ((e - s)a^3)^n = (e - s)a^3(e - s)a^3...(e - s)a^3 = (e - \bar{s})a^m$ for some m an some $\bar{s} \in \langle a \rangle$. The last equation involves a reduction technique which we will illustrate with $n = 3$.

$$0 = (e - s)a^3(e - s)a^3(e - s)a^3 = (e - s)^2 a^k (e - s)a^3.$$

Case (i) Assume $k \leq 3$. Since N has the IFP, then

$$0 = (e-s)^2 a^k a^{k-3}(e-s)a^3 = (e-s)^3 a^k.$$

Observe there exists $\bar{s} \in \langle a \rangle$ such that $(e-s)^3 = e-\bar{s}$. Hence $0 = (e-\bar{s})a^k$, so $a^k = \bar{s}a^k \in \langle a \rangle a^k$.

Case (ii) Assume $k > 3$. Let p be the least positive integer such that $k \leq 3p$. Since N has the IFP, then

$$0 = (e-s)^2 a^{3p}(e-s)a^3 = (e-s)^2 a^{3(p-1)} a^3 (e-s)a^3 =$$
$$= (e-s)^2 a^{3(p-1)}(e-s)a^k = (e-s)^2 a^{3(p-2)} a^3 (e-s) a^3 a^{3(p-1)} =$$
$$= (e-s)^2 a^{3(p-2)}(e-s)a^{k+3(p-1)} = \ldots = (e-s)^3 a^m = (e-\bar{s})a^m.$$

Hence $a^m \in \langle a \rangle a^m$.◊

From [6, Th. 4.4], we have that if every prime ideal of N is completely prime then every ideal is 2-primal (hence if $(0:a) \lhd N$, then $(0:a)$ is 2-primal). Examples of such near-rings with unity are provided in [6].

From [10], N satisfies the *CZ1 condition* if for any $x, y \in N$ and positive integer k such that $(xy)^k = 0$, then there exists a positive integer m such that $x^m y^m = 0$. Observe if $\mathcal{N}(N)$ is contained in the multiplicative center of N, then N satisfies the CZ1 condition. However [10, Example 2.5] provides an example of a ring R with unity such that R satisfies CZ1, but $\mathcal{N}(R)$ is not contained in the center of R.

Theorem 3.3. *Let N be an IFP near-ring with left unity e which satisfies the CZ1 condition, and $(0:a)$ is a 2-primal ideal for all $a \in N$. Then the following conditions are equivalent:*

(i) *N is left pseudo π-regular.*

(ii) *Every prime ideal is maximal.*

(iii) *Every completely prime ideal is maximal.*

(iv) *For every $a \in N$ there exists n, possibly depending on a, such that $N = (0:a^n) + \langle a \rangle$.*

Proof. (i) ⇒ (ii). This implication follows from Cor. 2.4.

(ii) ⇒ (iii). This implication is obvious.

(iii) ⇒ (i). Let $a \in N$. By Prop. 3.2(i), there exists a positive integer k and $s \in \langle a \rangle$ such that $0 = ((e-s)a^3)^k$. Since N satisfies the CZ1 condition, there exists a positive integer m such that $0 = (e-s)^m a^{3m}$. There exists $\bar{s} \in \langle a \rangle$ such that $(e-s)^m = e-\bar{s}$. Hence $a^{3m} = \bar{s}a^{3m}$. Therefore N is left pseudo π-regular.

(i) ⇔ (iv). This equivalence follows from Cor. 1.5.◊

Lemma 3.4. *If N is a reduced near-ring, and $0 \neq a \in N$, then $N/(0:a)$ is reduced and $\bar{a} \in N/(0:a)$ is not a divisor of zero.*

Proof. Let $0 \neq a \in N$. From [24, Prop. 9.3], it follows that $(0:a) \lhd N$. Let $x^n \in (0:a)$. Hence $x^n a = 0$. Since (0) is a completely semiprime ideal, it follows from [17, Lemma 2.1(ii)] that $xa = 0$. Hence

$x \in (0 : a)$ and it follows that $(0 : a)$ is a completely semi-prime ideal and consequently $N/(0 : a)$ is reduced. The element $\bar{a} \in N/(0 : a)$ is nonzero since N is reduced. Now suppose $\bar{b}\bar{a} = \bar{0}$. From [17, Lemma 2.1], it follows that $\bar{a}\bar{b} = \bar{0}$. Hence $aba = 0$ and, therefore, $(ba)^2 = 0$. Since N is reduced, we have $\bar{b} = \bar{0}$.◊

Lemma 3.5. *If N is a reduced near-ring, then N has the IFP and $(0 : a)$ is a 2-primal ideal of N for all $a \in N$.*

Proof. From [17, Lemma 2.1(ii)] N is IFP. By Lemma 3.4 $N/(0 : a)$ is reduced. Thus $(0 : a)$ is a 2-primal ideal of N.◊

Lemma 3.6. *A near-ring N with left unity e is reduced and left weakly regular if and only if $N = (0 : a) \bigoplus \langle a \rangle$ for every $a \in N$.*

Proof. Assume N is reduced and left weakly regular. Since N is reduced, 0 is a completely semiprime ideal. By [17, Lemma 2.1], $(0 : a^n) = (0 : a)$. From Cor. 1.6, we have $N = (0 : a) + \langle a \rangle$. We show this sum is direct. Let $x \in (0 : a) \cap \langle a \rangle$. Now $xa = 0$ and $x \in \langle a \rangle$. Since N is reduced, we can show that $x \langle a \rangle = 0$. Hence $x^2 = 0$ and since N is reduced, we have $x = 0$.

For the converse, suppose $N = (0 : a) \bigoplus \langle a \rangle$ for all $a \in N$. We first show N is reduced. Let $a \in N$ such that $a^2 = 0$. Now $a \in (0 : a) \cap \langle a \rangle = 0$. Next we show N is weakly regular. Let $a \in N$. Since $e \in N = (0 : a) \bigoplus \langle a \rangle$, we have $a = e \cdot a = (t_1 + t_2)a$ with $t_1 \in (0 : a)$ and $t_2 \in \langle a \rangle$. Hence $a = t_1 a + t_2 a = t_2 a \in \langle a \rangle a$.◊

As in [27], we define O_p to be $\{a \in N \mid ba = 0, \text{ for some } b \notin P\}$, where P is a prime ideal of N.

Observe if N is a reduced near-ring and e is a left unity, then e is a unity. To see this observe $(xe - x)^2 = xe(xe - x) - x(xe - x) = 0$. Hence $xe = x$.

The following corollary generalizes [23, Lemma 4 and Th. 2].

Corollary 3.7. *Let N be a reduced near-ring with unity. The following conditions are equivalent:*

(i) *N is left weakly regular.*

(ii) *N is left pseudo π-regular.*

(iii) *Every prime ideal of N is maximal.*

(iv) *Every completely prime ideal of N is maximal.*

(v) *For every $a \in N$ we have $N = (0 : a) \bigoplus \langle a \rangle$.*

(vi) *For each prime ideal P of N, $P = O_p$.*

Proof. The equivalence of parts (i) through (v) follows from Th. 3.3, Lemma 3.5, and Lemma 3.6.

(i) $\Rightarrow$ (vi). Let P be any prime ideal. Since (i) $\Leftrightarrow$ (iii) we have that N has the *pm* condition and form Cor. 2.7 we have P is completely prime. Let $x \in O_p$. Then there exists $b \notin P$ such that $bx = 0 \in P$. Now the fact that P is completely prime forces $x \in P$ and with $(1-a) \in N\backslash P$. Hence $x \in O_p$ and we have $O_p = P$.

(vi) $\Rightarrow$ (iii). Suppose $P = O_p$ for each P. Let M be a maximal ideal such that $P \subseteq M$. From [24, Cor. 2.72], M is also a prime ideal. Now from our assumption we have $M = O_M \subseteq O_p = P$.◊

Corollary 3.8. *Let N be a reduced near-ring with unity. N is left weakly regular if and only if every prime factor near-ring of N is a simple integral near-rings.*

Observe from [24, Th. 9.36] and Corrolary 3.8, we have: if N is a reduced left weakly regular near-ring with unity, then N is a subdirect product of simple integral near-rings.

Corollary 3.9. *Let N be a reduced near-ring with unity and DCC on N-subgroups. Then every prime factor near-ring of N is a near-field.*

Proof. The proof follows from Cor. 3.7, Prop. 1.3, and [24, Remarks 9.48d].◊

This corollary is in contrast to the statement after Prop. 1.3. An alternative proof of Cor. 3.9 can be given using [24, Prop. 9.41]. Observe there are finite simple reduced near-rings which are not near-fields [24, Remark 9.40].

Corollary 3.10. *Let N be a 2-primal near-ring with a left unity. The following conditions are equivalent:*

(i) *$N/\mathcal{P}_0(N)$ is left weakly regular.*

(ii) *$N/\mathcal{P}_0(N)$ is left pseudo π-regular.*

(iii) *Every prime ideal of N is maximal.*

(iv) *Every completely prime ideal of N is maximal.*

Proof. Since N is 2-primal, $N/\mathcal{P}_0(N)$ is a reduced near-ring. The remainder of the proof follows routinely from Cor. 3.7.◊

Observe that in [8] an example was given to show that the condition in Cor. 3.10 "N is 2-primal" is not superfluous. Also there are nonreduced nearrings satisfying the hypothesis of Th. 3.4. For example, let B be a reduced near-ring with unity and C is nonreduced commutative ring with unity. Then $N = B \bigoplus C$ satisfies the hypothesis of Th. 3.4.

Question. Is a reduced left weakly regular near-ring (with unity) also right weakly regular?

If we let $N_P = \{a \in N : ba \in \mathcal{P}_0(N)$ for some $b \in P\backslash N\}$ we can now characterize minimal prime ideals in reduced near-rings similar to that for rings in [27].

Theorem 3.11. *Let N be a reduced near-ring. If P is any prime ideal of N, then*

$$O_p = \cap\{Q \lhd N \mid Q \text{ prime and } O_p \subseteq Q\}$$
$$= \cap\{Q \lhd N \mid Q \text{ prime and } Q \subseteq P\}.$$

Proof. Let P be any prime ideal and suppose Q is a prime ideal such that $Q \subseteq P$. We first show that $O_Q \subseteq Q$. Let $z \in O_Q$. Now $bz = 0$ for some $b \notin Q$. Since $bz = 0$ and N reduced, it follows from [6, Lemma 2.5] that $\langle b\rangle \langle z\rangle = 0 \subseteq Q$. Since Q is a prime ideal and $b \notin Q$ we must have $z \in Q$ hence $O_Q \subseteq Q$ for any prime ideal Q. For $Q \subseteq P$ we now have $O_p \subseteq Q_Q \subseteq Q$ and consequently

$$O_p \subseteq \cap\{Q \mid O_p \subseteq Q\} \subseteq \cap\{Q \mid Q \subseteq P\}.$$

Suppose now $a \notin O_p$. We shall find a prime ideal Q such that $a \notin Q$ and $Q \subseteq P$. Let $S = \{a, a^2, a^3...\}$. S is a multiplicative system that does not contain 0. Let $L = N\backslash P$, i.e. L is an m-system. Let T be the set of all nonzero elements of N of the form $a^{t_0}x_1a^{t_1}x_2...a^{t_{n-1}}x_na^{t_n}$ where $x_i \in L$ and the t_i's are positive integers with t_0 and t_n allowed to be zero. Clearly $L \subseteq T$. Let $M = T \cup S$. We show that M is a m-system. Let $x, y \in M$. If $x, y \in S$ then $xy \in S \subseteq M$ and we are done. Let $x \in S$ and $y \in T$, say $x = a^s$ and $y = a^{t_0}y_1a^{t_1}y_2a^{t_2}...y_ma^{t_m}$. If $\langle x\rangle \langle y\rangle \neq 0$, then $xy \neq 0$. This follows form the fact that N is reduced and from the contrapositive of Lemma 2.5 of [6]. Since $xy \neq 0$ we have $xy \in T$, hence $\langle x\rangle \langle y\rangle \cap M \neq \varnothing$. We show $xy = 0$ is impossible. Suppose $xy = 0$, then we have $xy = a^s \cdot a^{t_0}y_1a^{t_1}y_2...a^{t_{m-1}}y_ma^{t_m} = 0$. Since 0 is a completely semi-prime ideal, Lemma 3.1 yields $xy = a^l y_1y_2...y_m = 0$, where $l = s + +t_0 + ... + t_m$. From [6, Lemma 2.5] it follows that $\langle a^l\rangle \langle y_1\rangle \langle y_2\rangle ... \langle y_m\rangle = 0$. Let $0 \neq w \in \langle y_1\rangle \langle y_2\rangle ... \langle y_m\rangle \cap L$. This is possible since L is an m-system. Now we have then $a^l w = 0$. Since N is reduced, we have $aw = 0$. Hence $wa = 0$ with $w \in N\backslash P$. So $a \in O_p$, a contradiction. So a similar argument can be used for $x \in T$ and $y \in S$ and for $x, y \in T$. This shows that M is an m-system disjoint from 0. From [17, Lemma 3.1] there exists a completely prime ideal Q disjoint from M. Hence $a \notin Q$ and $Q \subseteq P$, completing the proof.◊

We have the following corollary:

Corollary 3.12. *If N is a 2-primal near-ring and P is any prime ideal, then:*

(i) $N_p = \cap\{Q \mid Q$*is a prime ideal and* $Q \subseteq P\}$;

(ii) P *is a minimal prime ideal if and only if* $P = N_P$.

Proof. If N is 2-primal then $\mathcal{P}_0(N) = \mathcal{P}_c(N)$ and, therefore, $N/\mathcal{P}_0(N)$ is reduced. Hence part (i) follows from the previous theorem, and part (ii) is a consequence of part (i).◊

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