Mathematica Pannonica
10/2 (1999), 169–175

EXPONENTIATION OF RELATIONAL SYSTEMS WITH RESPECT TO STRONG HOMOMORPHISMS

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Received: June 1998

MSC 1991: 08 A 02, 08 A 55

Keywords: n-ary relational system, strong homomorphism, power, antitransitivity, n-ary partial algebra

Abstract: For any pair of n-ary relational systems G, H we consider a certain power, i.e., an n-ary relational system carried by the set of all homomorphisms of H into G. The subsystem of the power carried by the set of all strong homomorphisms of H into G is then taken as the power of G and H with respect to strong homomorphisms. The obtained binary operation of exponentiation of n-ary relational systems with respect to strong homomorphisms is studied and the results are applied to partial algebras.

The category of relational systems of a given (finite) arity with strong homomorphisms as morphisms does not have products in general. Therefore this category is not cartesian closed and hence it does not have function spaces. Thus, if we introduce an operation of exponentiation for relational systems of the same arity with respect to (i.e. carried by) strong homomorphisms, it will not have the well behaviour which is characteristic for function spaces. Nevertheless, it is desirable that the exponentiation have as decent behaviour as possible. To obtain such an exponentiation, we will start from the cartesian closed topological category (see [5]) of reflexive relational systems of a given arity with usual homomorphisms as morphisms. The subobjects of powers (i.e., of function spaces) in this category carried by the strong homomorphisms are then considered to be powers with respect to strong homomorphisms.

By an *n-ary relational system*, n a natural number, we understand a pair (X, ρ) where X is a set, the so called *carrier* of (X, ρ), and $\rho \subseteq X^n$ is a subset. As usual when working with n-ary relations, we will not consider the trivial case $n = 1$. Thus, we assume that $n \geq 2$.

Let $G = (X, \rho)$, $H = (Y, \sigma)$ be a pair of n-ary relational systems. A map $f : X \to Y$ is said to be a *homomorphism* of G into H if $(x_1, ..., x_n) \in \rho$ implies $(f(x_1), ..., f(x_n)) \in \sigma$, a *strong homomorphism* if, whenever $x_1, ..., x_{n-1} \in X$ and $y \in Y$, we have $(f(x_1), ..., f(x_{n-1}), y) \in \sigma$ if and only if there is an elemant $x' \in X$ with $(x_1, \dots, x_{n-1}, x') \in \rho$ and $f(x') = y$. Clearly, a map $f : X \to Y$ is a strong homomorphism of G into H if and only if f is a homomorphism of G into H having the property that for arbitrary elements $x_1, \dots, x_{n-1} \in X$ and $y \in Y$ such that $(f(x_1), \dots, f(x_{n-1}), y) \in \sigma$ there is an element $x' \in X$ with $(x_1, \dots, x_{n-1}, x') \in \rho$ and $y = f(x')$.

We denote by $\mathrm{Hom}(G, H)$ the set of all homomorphisms of G into H and by $[G, H]$ the set of all strong homomorphisms of G into H. If G, H are n-ary relational systems, then by an *isomorphism* of G onto H we understans any bijective homomorphism of G into H for which the inverse map is a homomorphism of H into G. We write $G \cong H$ if G and H are isomorphic, i.e., if there is an isomorphism of G onto H. An n-ary relational system (X, ρ) is said to be a *subsystem* of an n-ary relational system (Y, σ) if $X \subseteq Y$ and $\rho = \sigma \cap X^n$. Given a pair of n-ary relational systems G, H, we write $G \preccurlyeq H$ if G can be embedded into H, i.e., if there is a subsystem H' of H such that $G \cong H'$. We will denote by $\times$ the direct product of n-ary relational systems, i.e., for n-ary relational systems $G = (X, \rho)$ and $H = (Y, \sigma)$ we have $G \times H = (X \times Y, \tau)$ where $\tau \subseteq (X \times Y)^n$ is given by $((x_1, y_1), \dots, (x_n, y_n)) \in \tau$ if and only if $(x_1, \dots, x_n) \in \rho$ and $(y_1, \dots, y_n) \in \sigma$.

Definition 1. Let $G = (X, \rho)$, $H = (Y, \sigma)$ be n-ary relational systems. By the *power* of G and H we understand the n-ary relational system $(\mathrm{Hom}(H, G), \tau)$ where $\tau \subseteq (\mathrm{Hom}(H, G))^n$ is given by $(f_1, \dots, f_n) \in \tau$ if and only if the implication $(y_1, \dots, y_n) \in \sigma \Rightarrow (f_1(y_1), \dots, f_n(y_n)) \in \rho$ is satisfied for any $y_1, \dots, y_n \in Y$. The subsystem $([H, G], \tau \cap [H, G]^n)$

of the power $(\mathrm{Hom}(H,G),\tau)$ of G and H is called the *power of G and H with respect to strong homomorphisms.*

For any pair of n-ary relational systems G, H we denote by G^H and $G \diamond H$ the power and the power with respect to strong homomorphisms, respectively, of G and H. As usual, an n-ary relational system (X,ρ) is said to be reflexive if $(x_1,\dots,x_n) \in \rho$ whenever $x_1 = x_2 = \dots = x_n \in X$.

Remark 2. For n-ary relational systems the so called direct power is often considered in the literature - see e.g. [6]. The direct power of n-ary relational systems $G = (X,\rho)$ and $H = (Y,\sigma)$ is the n-ary relational system $(\mathrm{Hom}(H,G),\varepsilon)$ where $\varepsilon \subseteq (\mathrm{Hom}(H,G))^n$ is given by $(f_1,\dots,f_n) \in \varepsilon$ if and only if $(f_1(y),\dots,f_n(y)) \in \rho$ for each $y \in Y$. Especially, if G and H are ordered sets, then the direct power of G and H is nothing else that the Birkhoff's cardinal power [1],[2]. Let $(\mathrm{Hom}(H,G),\tau)$ and $(\mathrm{Hom}(H,G),\varepsilon)$ be the power and the direct power of $G = (X,\rho)$ and H, respectively. Clearly, $\tau \subseteq \varepsilon$ whenever H is reflexive. In [8] it is shown that $\varepsilon \subseteq \tau$ whenever G is diagonal, i.e., whenever for any $n \times n$-matrix over X whose all rows and all columns belong to ρ it is true that also the diagonal belongs to ρ. (Clearly, a binary relational system (X,ρ) is diagonal if and only if ρ is transitive.) Hence, the power and the direct power of n-ary relational systems G and H coincide whenever G is diagonal and H is reflexive.

It is well known that the category of n-ary relational systems and homomorphisms is cartesian closed - its powers are obtained from the powers given in Def. 1 by replacing the set $\mathrm{Hom}(H,G)$ with the set of all maps of H into G. The category of reflexive n-ary relational systems and homomorphisms is also cartesian closed and, moreover, it is topological (in the sense of [5]). It means that the category has powers which are function spaces. By [8], these powers coincide with the powers from Def. 1. Consequently, the so called first exponential law $G^{H\times K} \cong (G^H)^K$ is satisfied whenever G, H, K are reflexive n-ary relational systems. It can easily be seen that the law remains valid also when omitting the requirement of reflexivity of G. (Evenmore, the first exponential law is satisfied whenever only K is reflexive. But in this case the natural law $(G^H)^K \cong (G^K)^H$ need not be valid.) Unfortunately, for powers of (reflexive) n-ary relational systems with respect to strong homomorphisms the first exponential law is not fulfilled in general. We will find conditions under which the weaker law $G \diamond (H \times K) \preccurlyeq (G \diamond H) \diamond K$ is satisfied.

Definition 3. An n-ary relational system (X,ρ) is called *antitransitive* if for any $x_1,\dots,x_n,y \in X$ from $(x_1,\dots,x_n) \in \rho$, $(x_n,\dots,x_n,y) \in \rho$ and $(x_1,\dots,x_{n-1},y) \in \rho$ it follows that $x_n = y$.

Example 4. Clearly, a binary relational system (X, ρ) is antitransitive if $(x, y) \in \rho$, $(y, z) \in \rho$ and $(x, z) \in \rho$ imply $y = z$. For example, let $\mathbb{R}$ be the set of real numbers and let $\rho \subseteq \mathbb{R}^2$ be given as follows: $(x, y) \in \rho \Leftrightarrow x$ is rational and either $x = y$ or x is irrational with $x < y$. Then $(\mathbb{R}, \rho)$ is antitransitive.

Proposition 5. *Let G, H be n-ary relational systems. Then*

(1) *$G \diamond H$ is reflexive,*

(2) *$G \diamond H$ is antitransitive provided that G is antitransitive and H is reflexive.*

Proof. Let $G = (X, \rho)$, $H = (Y, \sigma)$ and $G \diamond H = ([H, G], \tau)$.

(1) Let $f \in [H, G]$ be an arbitrary element. As $f \in \operatorname{Hom}(H, G)$, there holds $(f(y_1), \dots, f(y_n)) \in \rho$ whenever $(y_1, \dots, y_n) \in \sigma$. Hence $(f, \dots, f) \in \tau$, i.e. $G \diamond H$ is reflexive.

(2) Assume that G is antitransitive and H is reflexive. Let $f_1, \dots, \dots, f_n, g \in [G, H]$ be elements such that $(f_1, \dots, f_n) \in \tau, (f_n, \dots, f_n, g) \in \tau$ and $(f_1, \dots, f_{n-1}, g) \in \tau$. Then, whenever $y \in Y$, we have $(f_1(y), \dots, f_n(y)) \in \rho$, $(f_n(y), \dots, f_n(y), g(y)) \in \rho$ and $(f_1(y), \dots, \dots, f_{n-1}(y), g(y)) \in \rho$. As G is antitransitive, $f_n(y) = g(y)$ for any $y \in Y$. Hence $f_n = g$, so that $G \diamond H$ is antitransitive. $\Diamond$

Theorem 6. *Let G, H, K be n-ary relational systems. If G is antitransitive and H, K are reflexive, then $G \diamond (H \times K) \preccurlyeq (G \diamond H) \diamond K$.*

Proof. Let $G = (X, \rho)$, $H = (Y, \sigma)$, $K = (Z, \varepsilon)$ and suppose that G is antitransitive and H, K are reflexive. Put $H \times K = (Y \times Z, \kappa)$, $G \diamond (H \times K) = ([H \times K, G], \lambda)$, $G \diamond H = ([H, G], \tau)$ and $(G \diamond H) \diamond K = ([K, G \diamond H], \mu)$. For an arbitrary element $f \in [H \times K, G]$ let $\varphi(f) : Z \to X^Y$ (where X^Y denotes the set of all maps of Y to X) be the map given by $\varphi(f)(z)(y) = f(y, z)$. Let $z \in Z$ and $(y_1, \dots, y_n) \in \sigma$ be arbitrary elements. Then $((y_1, z), \dots, (y_n, z)) \in \kappa$ (because K is reflexive), i.e. $(f(y_1, z), \dots, f(y_n, z)) = (\varphi(f)(z)(y_1), \dots, \varphi(f)(z)(y_n)) \in \rho$. Hence $\varphi(f)(z) \in \operatorname{Hom}(H, G)$. Let $y_1, \dots, y_{n-1} \in Y$ and $x \in X$ be elements having the property that $(\varphi(f)(z)(y_1), \dots, \varphi(f)(z)(y_{n-1}), x) \in \rho$. Then $(f(y_1, z), \dots, f(y_{n-1}, z), x) \in \rho$. As f is a strong homomorphism, there is an element $(y', z') \in Y \times Z$ such that $((y_1, z), \dots, (y_{n-1}, z), (y', z')) \in \kappa$ and $f(y', z') = x$. Since $(y_1, \dots, y_{n-1}, y') \in \sigma$ and $(z, \dots, z, z') \in \varepsilon$, we have $((y_1, z), \dots, (y_{n-1}, z), (y', z)) \in \kappa$ and $((y', z), \dots, (y', z), (y', z')) \in \kappa$ (because H and K are reflexive). Now from $f \in \operatorname{Hom}(H \times K, G)$ it follows that $(f(y_1, z), \dots, f(y_{n-1}, z), f(y', z)) \in \rho$, $(f(y', z), \dots, f(y', z), f(y', z')) \in \rho$ and $(f(y_1, z), \dots, f(y_{n-1}, z), f(y', z')) \in \rho$. Thus, the antitransitivity of G results in $f(y', z) = f(y', z') = x$, i.e. $\varphi(f)(z)(y') = x$. Therefore $\varphi(f)(z) \in [H, G]$. We have shown that $\varphi(f)$ maps Z into

$[H,G]$. Let $(z_1,\dots,z_n)\in\varepsilon$ be an arbitrary element. Then for any $(y_1,\dots,y_n)\in\sigma$ we have $((y_1,z_1),\dots,(y_n,z_n))\in\kappa$, hence $(f(y_1,z_1),\dots,$ $\dots,f(y_n,z_n))=(\varphi(f)(z_1)(y_1),\dots,\varphi(f)(z_n)(y_n))\in\rho$. Thus $(\varphi(f)(z_1),$ $\dots,\varphi(f)(z_n))\in\tau$. Consequently, $\varphi(f)\in\mathrm{Hom}(K,G\diamond H)$.

Let $z_1,\dots,z_{n-1}\in Z$ and $g\in[H,G]$ be arbitrary elements with $(\varphi(f)(z_1),\dots,\varphi(f)(z_{n-1}),g)\in\tau$. Let $y\in Y$ be an element. Then $(\varphi(f)(z_1)(y),\dots,\varphi(f)(z_{n-1})(y),g(y))=(f(y,z_1),\dots,f(y,z_{n-1}),g(y))\in$ $\in\rho$. As f is a strong homomorphism, there exists an element $(y',z')\in$ $\in Y\times Z$ with $((y,z_1),\dots,(y,z_{n-1}),(y',z'))\in\kappa$ and $f(y',z')=g(y)$. Since $(y,\dots,y,y')\in\sigma$ and $(z_1,\dots,z_{n-1},z')\in\varepsilon$, we have $((y,z_1),\dots,$ $\dots,(y,z_{n-1}),(y,z'))\in\kappa$ and $((y,z'),\dots,(y,z'),(y',z'))\in\kappa$ (because H and K are reflexive). Hence $(f(y,z_1),\dots,f(y,z_{n-1}),f(y,z'))\in\rho$, $(f(y,z'),\dots,f(y,z'),f(y',z'))\in\rho$ and $(f(y,z_1),\dots,f(y,z_{n-1}),f(y',z'))\in$ $\in\rho$. Now the antitransitivity of G implies $f(y,z')=f(y',z')=g(y)$, i.e. $\varphi(f)(z')(y)=g(y)$. Therefore $\varphi(f)(z')=g$. We have shown that $\varphi(f)\in[K,G\diamond H]$. Thus, φ maps $[H\times K,G]$ into $[K,G\diamond H]$ and it is evident that φ is an injection.

Let $(f_1,\dots,f_n)\in\lambda$, $(y_1,\dots,y_n)\in\rho$ and $(z_1,\dots,z_n)\in\varepsilon$ be arbitrary elements. Then $((y_1,z_1),\dots,(y_n,z_n))\in\kappa$ and we have

$$(\varphi(f_1)(z_1)(y_1),\dots,\varphi(f_n)(z_n)(y_n))=(f_1(y_1,z_1),\dots,f_n(y_n,z_n))\in\rho.$$

Thus $(\varphi(f_1)(z_1),\dots,\varphi(f_n)(z_n)(y_n))\in\tau$, which yields $(\varphi(f_1),\dots,\varphi(f_n))\in$ $\in\mu$. We have shown that $\varphi\in\mathrm{Hom}(G\diamond(H\times K),(G\diamond H)\diamond K)$. Further, let $f_1,\dots,f_n\in[H\times K,G]$ be arbitrary elements with $(\varphi(f_1),\dots,\varphi(f_n))\in\mu$, and let $(y_1,z_1),\dots,(y_n,z_n)\in\kappa$ be an element. Then $(y_1,\dots,y_n)\in\sigma$, $(z_1,\dots,z_n)\in\varepsilon$ and we have $(\varphi(f_1)(z_1),\dots,\varphi(f_n)(z_n))\in\tau$ and

$$(\varphi(f_1)(z_1)(y_1),\dots,\varphi(f_n)(z_n)(y_n))=(f_1(y_1,z_1),\dots,f_n(y_n,z_n))\in\rho.$$

Hence $(f_1,\dots,f_n)\in\lambda$. Therefore φ is an embedding of $G\diamond(H\times K)$ into $(G\diamond H)\diamond K$ and the proof is complete. $\Diamond$

Corollary 7. *Let G, H, K be binary relational systems. If G is antitransitive and H, K are reflexive, then $G\diamond(H\times K)\cong(G\diamond H)\diamond K$.*

Proof. It is sufficient to show that in the proof of Theorem 6 the map $\varphi:[H\times K,G]\to[K,G\diamond H]$ is a surjection for $n=2$. To this end, let $h\in[K,G\diamond H]$ be an arbitrary element and put $h^*(y,z)=$ $=h(z)(y)$ for any $y\in Y$ and $z\in Z$. Let $((y_1,z_1),(y_2,z_2))\in\kappa$. Then $(z_1,z_2)\in\varepsilon$, hence $(h(z_1),h(z_2))\in\tau$. Thus, since $(y_1,y_2)\in\sigma$, we have $(h(z_1)(y_1),h(z_2)(y_2))\in\rho$, i.e. $(h^*(y_1,z_1),h^*(y_2,z_2))\in\rho$. Therefore $h^*\in\mathrm{Hom}(H\times K,G)$. Let $(y,z)\in Y\times Z$ and $x\in X$ be elements such that $(h^*(y,z),x)\in\rho$. Then $(h(z)(y),x)\in\rho$ and, as $h(z)\in[G,H]$, there exists $y'\in Y$ such that $(y,y')\in\sigma$ and $h(z)(y')=x$. But then

$((y,z),(y',z)) \in \kappa$ and $h^*(y',z) = x$. Consequently, $h^* \in [H \times K, G]$. As clearly $\varphi(h^*) = h$, the proof is complete. $\Diamond$

Remark 8. a) A reflexive n-ary relational system (X,ρ) is antitransitive if and only if for any $x, y \in X$ there holds $(x, \dots, x, y) \in \rho \Rightarrow x = y$. Especially, a reflexive binary relational system (X,ρ) is antitransitive if and only if it is discrete (i.e. if and only if ρ is the equality). Thus, by Prop. 5, the relational systems $G \diamond (H \times K)$ and $(G \diamond H) \diamond K$ in Cor. 7 are discrete.

b) Clearly, if G, H, K are n-ary relational systems, then the first exponential law $G \diamond (H \times K) \cong (G \diamond H) \diamond K$ is trivially satisfied whenever the carrier of G is a singleton.

c) For well-behaved exponentiations also the so-called second and third exponential laws are valid. Especially, for exponentiation of n-ary relational systems with respect to strong homomorphisms these laws have the forms

(1) $(\underset{i\in I}{\rightarrow} \prod G_i) \diamond H \cong \underset{i\in I}{\rightarrow} \prod (G_i \diamond H)$ and

(2) $G \diamond (\underset{i\in I}{\rightarrow} \coprod H_i) \cong \underset{i\in I}{\rightarrow} \prod (G \diamond H_i)$

(where $\prod$ and $\coprod$ denote the direct product and the direct sum). As projections of direct products of n-ary relational systems need not be strong homomorphisms, the second exponential law (1) is not valid in general. So, it is an open problem to find conditions under which the law (1), or a weaker form of it, is satisfied. On the other hand, the third exponential law (3) is valid because canonical injections into direct sums of n-ary relational systems are strong homomorphisms.

In the rest of the paper we will deal with applications of the previous outcomes to partial algebras.

An n-ary partial algebra (n a natural number) is an $(n+1)$-ary relational system (X,ρ) such that, whenever $x_1, \dots, x_n, y, z \in X$, from $(x_1, \dots, x_n, y) \in \rho$ and $(x_1, \dots, x_n, z) \in \rho$ it follows that $y = z$. Thus, it is evident that each n-ary partial algebra is antitransitive. Reflexive partial algebras are usually called *idempotent* and some authors (see e.g. [3]) speak about closed homomorphisms istead of about strong homomorphisms of partial algebras.

Remark 9. An n-ary *algebra* is an n-ary partial algebra (X,ρ) with the property that for any $x_1, \dots, x_n \in X$ there exists an element $y \in Y$ such that $(x_1, \dots, x_n, y) \in \rho$. Let G be an n-ary algebra and H be an n-ary partial algebra. If $[H,G] \neq \emptyset$, then $G \diamond H = G^H$. This follows from the fact that, whenever there exists a strong homomorphism of H into G, also H is an n-ary algebra (and hence each homomorphism of H into G is strong).

Proposition 10. *Let G be an n-ary partial algebra and H be a reflexive $(n+1)$-ary relational system. Then $G \diamond H$ is an idempotent n-ary partial algebra.*

Proof. Let $G = (X, \rho)$, $H = (Y, \sigma)$, $G \diamond H = ([H, G], \tau)$ and let $f_1, \dots, f_n, g, h \in [H, G]$. Let $(f_1, \dots, f_n, g) \in \tau$, $(f_1, \dots, f_n, h) \in \tau$ and let $y \in Y$ be an arbitrary element. Since $(y, \dots, y) \in \sigma$, we have $(f_1(y), \dots, f_n(y), g(y)) \in \rho$ and $(f_1(y), \dots, f_n(y), h(y)) \in \rho$. Consequently, $g(y) = h(y)$. Hence $g = h$, so that $G \diamond H$ is a partial algebra. The idempotency of $G \diamond H$ follows from Prop. 5. $\Diamond$

Th. 6 and Prop. 10 result in

Corollary 11. *Let G be an n-ary partial algebra and H, K be reflexive $(n+1)$-ary relational systems. Then $G \diamond (H \times K)$ and $(G \diamond H) \diamond K$ are idempotent n-ary partial algebras with $G \diamond (H \times K) \preccurlyeq (G \diamond H) \diamond K$.*

Remark 12. a) By Cor. 7, if G is an unary partial algebra and H, K are reflexive binary relational systems, then $G \diamond (H \times K)$ and $(G \diamond H) \diamond K$ are isomorphic idempotent unary algebras.

b) The problem of the validity of the first exponential law $G^{H \times K} \cong (G^H)^K$ for powers of n-ary partial algebras with respect to homomorphisms is dealt with in [7] where some cartesian closed subcategories of the category of n-ary partial algebras and homomorphisms are found.

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