

CONTRA-SEMICONTINUOUS FUNCTIONS

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Abstract: The aim of this paper is to introduce and study the concept of a contra-semicontinuous function and further investigate the class of strongly S -closed spaces. We obtain some new decompositions of generalized continuous functions.

1. Introduction

Covering spaces with closed sets has its historical background in General Topology.

In 1918, Sierpinski [19] proved that if a connected compact Hausdorff space has a countable cover of pairwise disjoint closed sets, at most one of those sets is nonvoid. In 1992, Cater and Daily [5] showed that if a complete, connected, locally connected metric space is covered by

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countably many proper closed sets, then some two members of these sets must meet. Cater and Daily improved slightly Sierpinski's result by proving that some two members must meet in at least continuum many points. Their new result has applications to several spaces frequently encountered in functional analysis (see [5, Corollaries 2,3 and 4]). In 1996, the first author [7] considered spaces where every cover by closed sets has a finite subcover. Such spaces are called *strongly S -closed*. This concept generalizes Thompson's [22] S -closed spaces, whose definition requires that every cover by regular closed sets has a finite subcover. It is a natural to ask which class of generalized continuity 'transforms' strongly S -closed spaces onto compact spaces. Such functions are called in [7] *contra-continuous*. In this paper we consider a slightly weaker form of contra-continuity called *contra-semicontinuity* and study in detail its properties. We further investigate the class of strongly S -closed spaces and obtain several decompositions of generalized continuous functions.

2. Preliminaries

A subset A of a topological space (X, τ) is called *semi-regular* [6] if A is both semi-open and semi-closed. If A is the intersection of an open set and a semi-closed (resp. semi-regular) set, then A is called a \mathcal{B} -set [23] (resp. \mathcal{AB} -set [8]). The *semi-closure* of A , denoted by $sCl(A)$, is the intersection of all semi-closed supersets of A . If $sCl(A) \subseteq U$ whenever U is open (resp. semi-open) and $A \subseteq U$, then A is called *gs-closed* [2] (resp. *sg-closed* [3]). Recall additionally that a set A is called *simply-open* [16] if $A = U \cup N$, where U is open and N is nowhere dense.

The definitions of some basic concepts such as semi-open set, semi-continuous function, α -open set, etc. can be found in many papers, for instance [7].

The family of all semi-open (resp. semi-closed, semi-regular, regular open, regular closed, α -open, preopen, β -open, clopen) subsets of a topological space (X, τ) will be denoted by $SO(X)$ (resp. $SC(X)$, $SR(X)$, $RO(X)$, $RC(X)$, $\alpha(X)$, $PO(X)$, $\beta(X)$, $CO(X)$).

Lemma 2.1. *For a subset A of a space X , the following conditions are equivalent:*

- (1) $A \in SR(X)$.
- (2) $A \in \beta(X) \cap SC(X)$.

Lemma 2.2. *For a subset A of a space (X, τ) , the following conditions are equivalent:*

- (1) $A \in RO(X)$.

- (2) $A \in \tau \cap SC(X)$.
- (3) $A \in \alpha(X) \cap SC(X)$.
- (4) $A \in PO(X) \cap SC(X)$.

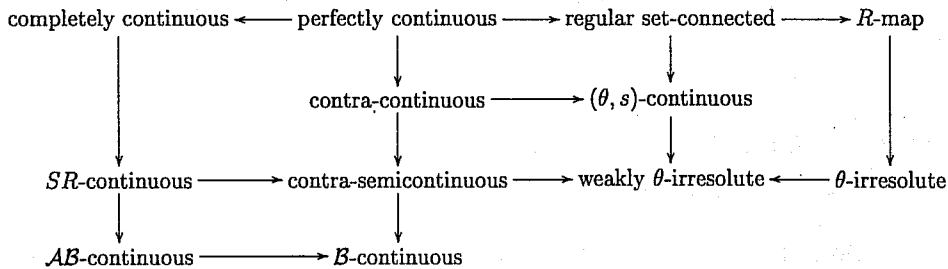
A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called *perfectly continuous* [17] (resp. *completely continuous* [1], *SR-continuous*, *RC-continuous*, *B-continuous* [23], *AB-continuous* [8], *simply-continuous* [16]) if the preimage of every open subset of Y is clopen (resp. regular open, semi-regular, regular closed, \mathcal{B} -set, \mathcal{AB} -set, simply-open) in X .

A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called *regular set-connected* [9] (resp. (θ, s) -continuous [14], *weakly θ -irresolute* [13], *R-map* [4], θ -irresolute [15]) if the preimage of every regular open subset of Y is clopen (resp. closed, semi-closed, regular open, intersection of regular open sets) in X .

3. Contra-semicontinuous functions

Definition 1. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called *contra-semicontinuous* (resp. *contra-continuous* [7]) if the preimage of every open subset of Y is semi-closed (resp. closed) in X .

The following diagram shows how contra-semicontinuous functions are related to some similar types of generalized continuity.



The following examples show that contra-semicontinuity is placed strictly between contra-continuity and \mathcal{B} -continuity as well as strictly between SR -continuity and weakly θ -irresolute continuity.

Example 3.1. A contra-semicontinuous function need not be contra-continuous. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function $f(x) = [x]$, where $[x]$ is the Gaussian symbol. If V is a closed subset of the real line, its preimage $U = f^{-1}(V)$ is the union of intervals of the form $[n, n + 1)$, $n \in \mathbb{Z}$; hence U is semi-open being union of semi-open sets. But f is not contra-continuous, since $f^{-1}(0.5, 1.5) = [1, 2)$ is not closed in \mathbb{R} .

Example 3.2. The identity function on the real line (with the usual topology) is \mathcal{B} -continuous but not contra-semicontinuous, since the preimage of each singleton fails to be semi-open.

Example 3.3. A contra-semicontinuous function need not be SR -continuous. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{c\}, X\}$. The identity function $f: (X, \tau) \rightarrow (X, \sigma)$ is even contra-continuous but not SR -continuous, since $A = \{a, b\} \in \sigma$ but A is not semi-regular in (X, τ) .

Example 3.4. A weakly θ -irresolute function need not be contra-semicontinuous. Let $X = \{a, b\}$, $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. The identity function $f: (X, \tau) \rightarrow (X, \sigma)$ is weakly θ -irresolute as only the trivial subsets of X are regular open in (X, σ) . However, $f^{-1}(\{a\}) = \{a\}$ is not semi-closed in (X, τ) ; hence f is not contra-semicontinuous.

Proposition 3.5. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is contra-semicontinuous.
- (2) For every closed subset F of Y , $f^{-1}(F) \in SO(X, \tau)$.
- (3) For each $x \in X$ and each closed subset F of Y containing $f(x)$, there exists a semi-open $U \in SO(X, \tau)$ such that $f(U) \subseteq F$.
- (4) $\text{Int}(\text{Cl}(f^{-1}(V))) = \text{Int}(f^{-1}(V))$ for every $V \in \sigma$.
- (5) $\text{Cl}(\text{Int}(f^{-1}(F))) = \text{Cl}(f^{-1}(F))$ for every closed set F of Y .

Next, we offer the following three decomposition theorems.

Theorem 3.6. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

- (1) f is SR -continuous.
- (2) f is β -continuous and contra-semicontinuous.

Proof. Follows directly from Lemma 2.1. \diamond

Corollary 3.7. [7, Th. 3.11] Every contra-continuous, β -continuous is semi-continuous.

Example 3.8. The concepts of β -continuity and contra-semicontinuity are independent from each other. Consider the classical Dirichlet function $f: \mathbb{R} \rightarrow \mathbb{R}$, where \mathbb{R} is the real line with the usual topology:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily observed that f is β -continuous (in fact, f is even pre-continuous). But f is neither contra-continuous nor SR -continuous as \mathbb{Q} is not semi-closed (hence not semi-regular).

Theorem 3.9. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

- (1) f is completely continuous.
- (2) f is precontinuous and contra-semicontinuous.

Proof. Follows directly from Lemma 2.2. \diamond

Example 3.10. The identity function on real line \mathbb{R} with the usual topology is (pre)continuous but it is neither contra-semicontinuous nor completely continuous. For example, $f^{-1}(\mathbb{R} \setminus \{0\})$ is neither semi-closed nor regular open.

Theorem 3.11. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

- (1) f is RC -continuous.
- (2) f is β -continuous and contra-continuous.

Proof. Follows easily from the proof of [7, Th. 3.10]. \diamond

Remark 3.12. The classical Dirichlet function from Ex. 3.8 shows that a β -continuous function need not be either RC -continuous nor contra-continuous. On the other hand, Ex. 3.3 shows that a contra-continuous function need not be β -continuous.

Definition 2. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called *contra-gs-continuous* (resp. *contra-sg-continuous*) if the preimage of every open subset of Y is gs-closed (resp. sg-closed) in X .

Theorem 3.13. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

- (1) f is contra-semicontinuous.
- (2) f is \mathcal{B} -continuous and contra-gs-continuous.

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (1): Let $V \subseteq Y$ be open. Set $f^{-1}(V) = U \cap F$, where $U \in \tau$ and F is semi-closed in (X, τ) . Clearly, $f^{-1}(V) \subseteq U$, $U \in \tau$. Hence, $sCl(f^{-1}(V)) \subseteq U$, since $f^{-1}(V)$ is gs-closed as f is contra-gs-continuous. Now, $Int(Cl(f^{-1}(V))) = Int(Cl(U \cap F)) \subseteq Int(Cl(U) \cap Cl(F)) = Int(Cl(U)) \cap Int(Cl(F)) \subseteq Int(Cl(U)) \cap F$, since F is semi-closed. So, $Int(Cl(f^{-1}(V))) \cap U \subseteq Int(Cl(U)) \cap U \cap F$. Since

$$Int(Cl(f^{-1}(V))) \cup f^{-1}(V) = sCl(f^{-1}(V)) \subseteq U$$

and $U \subseteq Int(Cl(U))$, we have $Int(Cl(f^{-1}(V))) \subseteq U \cap F = f^{-1}(V)$. This shows that $f^{-1}(V)$ is semi-closed. \diamond

The following example will show that the concepts of \mathcal{B} -continuity and contra-gs-continuity are independent from each other and that contra-gs-continuity is strictly weaker than contra-semicontinuity.

Example 3.14. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, X\}$. The identity function on (X, τ) is continuous and hence \mathcal{B} -continuous but not contra-gs-continuous, because $sCl\{a\} = X \not\subseteq \{a\} \in \tau$. Additionally,

the identity function $f: (X, \sigma) \rightarrow (X, \tau)$ is contra-g \mathcal{S} -continuous but not \mathcal{B} -continuous, because $f^{-1}(\{a\}) = \{a\}$ is not a \mathcal{B} -set in (X, σ) .

The following two results give a tridecomposition of SR -continuity and complete continuity.

Corollary 3.15. *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:*

- (1) f is SR -continuous.
- (2) f is β -continuous, \mathcal{B} -continuous and contra-g \mathcal{S} -continuous.

Proof. Follows from Th. 3.6 and Th. 3.13. \diamond

Corollary 3.16. *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:*

- (1) f is completely continuous.
- (2) f is precontinuous, \mathcal{B} -continuous and contra-g \mathcal{S} -semicontinuous.

Proof. Follows from Th. 3.9 and Th. 3.13. \diamond

Remark 3.17. In both corollaries stated above the three functions in conditions (2) are pairwise independent. The function $f: (X, \tau) \rightarrow (X, \tau)$ (resp. $f: (X, \sigma) \rightarrow (X, \tau)$) in Ex. 3.14 is precontinuous but not contra-g \mathcal{S} -continuous (resp. not \mathcal{B} -continuous).

Theorem 3.18. *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:*

- (1) f is contra-semicontinuous.
- (2) f is simply-continuous and contra-g \mathcal{S} -continuous.

Proof. It is similar to the proof of Th. 3.13 and hence omitted. \diamond

Example 3.19. Not every simply-continuous function is contra-g \mathcal{S} -continuous. Consider the following function $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \\ 0, & x = 0. \end{cases}$$

It can be easily observed that f is simply-continuous. But f is not contra-g \mathcal{S} -continuous, since $\{0\}$ is closed and its preimage $\{0\}$ is not semi-open.

Example 3.20. Ex. 3.4 shows that not every contra-g \mathcal{S} -continuous and precontinuous function is simply-continuous. In Ex. 3.4, $f: (X, \tau) \rightarrow (X, \sigma)$ is precontinuous but not contra-g \mathcal{S} -continuous.

Corollary 3.21. *For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:*

- (1) f is SR -continuous.
- (2) f is β -continuous, simply-continuous and contra-g \mathcal{S} -continuous.

Corollary 3.22. For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

- (1) f is completely continuous.
- (2) f is precontinuous, simply-continuous and contra-sg-semicontinuous.

Remark 3.23. The composition of even two contra-continuous functions need not be contra-semicontinuous. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{c\}, X\}$ and $\mu = \{\emptyset, \{a, b\}, X\}$. Let $f: (X, \tau) \rightarrow (X, \sigma)$ and $g: (X, \sigma) \rightarrow (X, \mu)$ be the identity functions. Note that both f and g are contra-continuous but their composition $g \circ f$ is not even contra-semicontinuous, since $\{c\}$ is closed in μ but $(g \circ f)^{-1}(\{c\}) \notin SO(X, \tau)$.

4. Strongly S-closed spaces

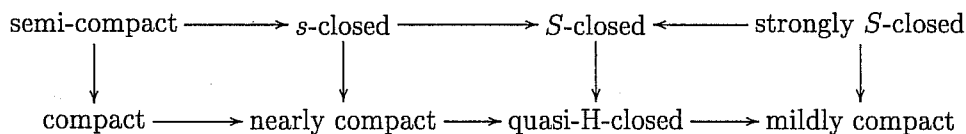
Definition 3. A topological space (X, τ) is called: (1) *semi-compact* [4] (resp. *s-closed* [6], *S-closed* [22]) if for every semi-open cover $\{V_i: i \in I\}$ of X , there exists a finite subset F of I such that $X = \cup\{V_i: i \in F\}$ (resp. $X = \cup\{sCl(V_i): i \in F\}$, $X = \cup\{Cl(V_i): i \in F\}$),

(2) *nearly compact* [20] (resp. *quasi-H-closed* [18]) if for every open cover $\{V_i: i \in I\}$ of X , there exists a finite subset F of I such that $X = \cup\{Int(Cl(V_i)): i \in F\}$ (resp. $X = \cup\{Cl(V_i): i \in F\}$),

(3) *strongly S-closed* [7] (resp. *mildly compact* [21]) if every closed (resp. clopen) cover of X has a finite subcover.

Lemma 4.1. A space X is *s-closed* (resp. *S-closed*, *nearly compact*) if and only if every *semi-regular* (resp. *regular closed*, *regular open*) cover of X has a finite subcover.

The implications in the following diagram are well-known.



Theorem 4.2. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a surjection. If one of the following conditions holds, then Y is strongly S-closed.

- (1) f is contra-semicontinuous and X is semi-compact,
- (2) f is SR-continuous and X is s-closed,
- (3) f is completely continuous and X is S-closed,
- (4) f is contra-continuous and X is compact,

- (5) f is RC -continuous and X is nearly compact,
 (6) f is perfectly continuous and X is mildly compact.

Proof. We will prove only the last condition, since proofs of the other ones are analogous. Let $\{V_i: i \in I\}$ be a closed cover of Y . Since f is perfectly continuous, $\{f^{-1}(V_i): i \in I\}$ is a clopen cover of X . Clearly, there exists a finite $F \subseteq I$ such that $X = \cup_{i \in F} f^{-1}(V_i)$ as X is mildly compact. Hence, $Y = \cup_{i \in F} V_i$. This shows that Y is strongly S -closed. \diamond

5. Some miscellaneous results

Theorem 5.1. *Let (X, τ) be connected and (Y, σ) be T_1 . If $f: (X, \tau) \rightarrow (Y, \sigma)$ is contra-continuous, then f is constant.*

Proof. We assume that Y is nonempty. Since Y is a T_1 -space, then $\mathcal{U} = \{f^{-1}(\{y\}): y \in Y\}$ is a disjoint open partition of X . If $|\mathcal{U}| \geq 2$, then there exists a proper nonempty set W (namely, some $U \in \mathcal{U}$). Since X is connected, then $|\mathcal{U}| = 1$. Hence, f is constant. \diamond

Corollary 5.2. *The only contra-continuous function defined on the real line \mathbb{R} is the constant one.*

Recall that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called *preclosed* [11] if the image of every closed subset of X is preclosed in Y . In 1969, El'kin defined a topological space (X, τ) to be *globally disconnected* [12] if every semi-open set is open. A space X is called *locally indiscrete* if every open set is closed.

Theorem 5.3. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a contra semi-continuous and pre-closed surjection. If X is globally disconnected, then Y is locally indiscrete.*

Proof. Let $V \in \sigma$. Since f is contra-semicontinuous, $f^{-1}(V) = U$ is semi-closed in X . Hence U is closed, since X is globally disconnected. Thus, $f(U) = V$ is preclosed in Y as f is pre-closed. Now $\text{Cl}(V) = \text{Cl}(\text{Int}(V)) \subseteq V$, i.e., V is closed. This shows that Y is locally indiscrete. \diamond

Theorem 5.4. *Contra-semicontinuous images of hyperconnected spaces are connected.*

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be contra-semicontinuous such that X is hyperconnected, i.e., every open subset of X is dense. Assume that B is a proper clopen subspace of Y . Then $A = f^{-1}(B)$ is both semi-open and semi-closed as f is contra-semicontinuous. This shows that A is semi-regular. Hence, $\text{Int}(A)$ and $\text{Int}(X \setminus A)$ are disjoint nonempty open subsets of X . This clearly contradicts with the fact that X is hyperconnected. Thus, Y is connected. \diamond

For a topological space X , the Cantor-Bendixson derivative $D(X)$ is the set of all non-isolated points of X . A topological space (X, τ) is called *sporadic* [10] if the Cantor-Bendixson derivative of X is meager. Recall also that a space X is called a $T_{\frac{1}{2}}$ -space if every singleton is open or closed.

Theorem 5.5. *If (X, τ) is CCC (= countable chain condition), Y is a $T_{\frac{1}{2}}$ -space and $f: (X, \tau) \rightarrow (Y, \sigma)$ is contra-semicontinuous, then Y is sporadic.*

Proof. If Y is discrete, then we are done. Assume next that $D(Y) \neq \emptyset$. Clearly, $\mathcal{U} = \{f^{-1}(\{y\}) : y \in D(Y)\}$ is nonempty. Since f is contra-semicontinuous, \mathcal{U} is a disjoint family of nonempty semi-open subsets of X . Then $\mathcal{V} = \{\text{Int}(f^{-1}(\{y\})) : y \in D(Y)\}$ is a disjoint family of nonempty open subsets of X . Since X is CCC, $|\mathcal{V}| \leq \aleph_0$. Hence, $|D(X)| \leq \aleph_0$. Since every singleton in $D(X)$ is nowhere dense, $D(X)$ is meager. This shows that Y is sporadic. \diamond

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