Mathematica Pannonica
10/2 (1999), 153–158

ON THE ITERATES OF THE SUM OF EXPONENTIAL DIVISORS

I. **Kátai**

Department of Computer Algebra, Eötvös L. University, H–1117 Budapest, Pázmány Péter sétány I/D, Hungary

M.V. **Subbarao**

Department of Mathematical Sciences, University of Alberta, Edmonton, T6G 2G1, Canada

Dedicated to Professor Ferenc Schipp on his 60th birthday

Received: November 1998

MSC 1991: 11 N 25, 11 N 60

Keywords: Exponential divisors, multiplicative function, distribution function.

Abstract: It is proved that if k is a fixed positive integer, then

$$\lim_{x\to\infty} \frac{1}{x}\#\{\, n\le x \mid \frac{f_j(n)}{f_{j-1}(n)} < \alpha_j,\ j=1,\dots,k \,\} = F_k(\alpha_1,\dots,\alpha_k)$$

exists and F_k is strictly monotonic in each variables in $(1,\infty)^k$, where $f_j(n)$ denotes the j-th iterate of the sum of exponential function $\sigma^{(e)}(n)$ defined as the multiplicative function which on prime powers p^a takes the value $\sigma^{(e)}(p^a) = \sum_{b|a} p^b$.

1. The sum of exponential functions $\sigma^{(e)}(n)$ is defined as that multiplicative function which on prime powers p^a takes the value

Supported by the grant D-41/97 for the first author and in part by an NSERC Grant for the second author.

$$\sigma^{(e)}(p^a) = \sum_{b|a} p^b,$$

where b runs over the positive divisors of a. It is clear that $\sigma^{(e)}(n) = n$ if $n = 1$ or square-free, and $\sigma^{(e)}(n) > n$ for the other integers.

Let $\sigma_k^{(e)}(n)$ be the k-th iterate of $\sigma^{(e)}(n)$; also let $\sigma_k(n)$, $\sigma_k^*(n)$ respectively denote the k-th iterates of $\sigma(n)$ - the sum of the positive divisors of n, and $\sigma^*(n)$ - the sum of the unitary divisors of n. We are interested in the limit distribution of the qualities $\sigma_k^{(e)}(n)/n$. Some of the properties of $\sigma_k^{(e)}(n)$ were earlier studied in [5] and [6]. We may recall that while $\sigma_2(n)/\sigma_1(n) \to \infty$ on a set of density unity (Erdős [1]), we have $\sigma_2^*(n)/\sigma_1^*(n) \to 1$ on a set of density unity (Erdős and Subbarao [6]). Also Kátai and M. Wijsmuller [4] have recently showed that $\frac{\sigma_3^*(n)}{\sigma_2^*(n)} \to 1$ on a set of density 1 and believe that the same holds for $\sigma_{r+1}^*(n)/\sigma_r^*(n)$ for all $r > 2$ also.

In [6] several questions were raised and many of these are still open. One of these questions is whether it is true that, on a set of density one, $\sigma_2^{(e)}(n)/\sigma_1^{(e)}(n) \to 1$. In response to this, Erdős proved [2] that this is not so. Actually, if S denotes the set given by $\{2^3 \cdot 5 \cdot \theta\}$, where θ varies over all odd square-free numbers relatively prime to 5, clearly S has a positive density. A simple computation shows that for any $m \in S$, we have $\sigma_1^{(e)}(m) = 2 \cdot 5^2 \cdot \theta$ while $\sigma_2^{(e)}(m) = 2 \cdot 30 \cdot \theta$, so that $\sigma_2^{(e)}(m)/\sigma_1^{(e)}(m) = 6/5$.

Erdős also stated without proof that it is possible to show that $\sigma_2^{(e)}(n)/\sigma_1^{(e)}(n)$ is dense in $(1, \infty)$ and has a distribution function which is everywhere monotone.

In this paper, we prove a more general result (see Th. 1).

2. To simplify the notation, we write $f_1(n) = \sigma^{(e)}(n)$, and $f_j(n) = \sigma_j^{(e)}(n)$, $f_o(n) = n$. Let $\mathcal{P}$ be the whole set of the primes. The letters p, q, π, ρ with and without suffixes always denote primes.

Our purpose in this paper is to prove the following

Theorem 1. *Let $k \geq 1$ be a fixed integer. Then*

$$(2.1)\quad \lim_{x\to\infty} \frac{1}{x} \#\{\, n \leq x \mid \frac{f_j(n)}{f_{j-1}(n)} < \alpha_j,\ j = 1, \ldots, k \,\} = F_k(\alpha_1, \ldots, \alpha_k)$$

exists, F_k is strictly monotonic in each variables in $(1, \infty)^k$.

Proof. The existence of the limit (2.1) is obvious. Even we can give an explicit form of F_k. Let $S = S_k$ be the set of those integers n for which the following assertion holds: for every prime divisor p of n for which $p^2 \nmid n$, there exists some $j \in \{1, \dots, k\}$ such that $p | f_j(\frac{n}{p})$. Each squarefull integer is considered to be an element of S as well.

The set S_1 is simple. We can write n as $K \cdot m$, where K is squarefull, m is square-free, and $(K, m) = 1$. Let $m = q_1, \dots, q_r$. Then $n \in S_1$ if and only if $q_i | f(K) f(\frac{m}{q_i})$ $(i = 1, \dots, r)$. Let $K(n)$ be the product of those primes π for which $\pi | f_1(n) f_2(n) \dots f_k(n)$, or what is the same, the product of the distinct prime divisors of $f_k(n)$.

Each integer N can be uniquely factorized in the form $N = n \cdot h$, where $n \in S_k$ and $(h, K(n)) = 1$, h square-free. We have $f_j(N) = f_j(n) h$ $(j = 0, \dots, k)$, i. e.

$$\frac{f_j(N)}{f_{j-1}(N)} = \frac{f_j(n)}{f_{j-1}(n)} \quad (j = 1, \dots, k). \tag{2.2}$$

Thus the values on the left hand side do depend only on the S-component of n.

Let $\kappa(p) := \frac{1}{1+1/p}$ $(p \in \mathcal{P})$, $\kappa(m)$ be strongly multiplicative. For some integer A let $M(x|A)$ be the number of those square-free integers $m \le x$ which are coprime to A. Then

$$M(x|A) = (1 + o(1)) \frac{6}{\pi^2} \kappa(A) x \quad (x \to \infty)$$

holds uniformly at least in the range $A \ll (\log x)^c$, say.

Let $\alpha_j > 1$ $(j = 1, \dots, k)$ be arbitrary real numbers. Let J be the set of those $n \in S_k$ for which

$$\frac{f_j(n)}{f_{j-1}(n)} < \alpha_j \quad (j = 1, \dots, k).$$

We get almost immediately that

$$F_k(\alpha_1, \dots, \alpha_k) = \frac{6}{\pi^2} \sum_{n \in J} \frac{\kappa(K(n))}{n}. \tag{2.3}$$

To prove (2.3), it is enough to observe that

$$\lim_{x \to \infty} \sup \frac{1}{x} \#\{ N \le x,\ S \text{ part of } N > H \} \to 0$$

as $H \to \infty$.

From (2.2) and the existence of the limit we get

Lemma 1. *For every $m > 0$,*

$$\lim_{x \to \infty} \frac{1}{x} \#\{ N \le x \mid \frac{f_j(N)}{f_{j-1}(N)} = \frac{f_j(m)}{f_{j-1}(m)},\ j = 1, \dots, k \}$$

exists, and it is positive.

Thus, for the strict monotonity of F_k it is enough to prove

Lemma 2. *Let $\alpha_1, \dots, \alpha_k > 1$, $\varepsilon > 0$ be arbitrary real numbers. Then there exists an integer n for which*

$$\alpha_j \le \frac{f_j(n)}{f_{j-1}(n)} < \alpha_j + \varepsilon \quad (j = 1, \dots, k).$$

3. Proof of Lemma 2. Let $p(n)$ be the smallest, $P(n)$ be the largest prime factor of n. Let $B_1, B_2, \dots, B_k$ be arbitrary positive even numbers, A_k be an arbitrary positive number. Let $A_{k-1} = f(A_k)B_k$, $A_{k-2} = f(A_{k-1})B_{k-1}, \dots, A_1 = f(A_2)B_2$, $Y > B_1 \dots B_k A_1 \dots A_k$. Let Y be an arbitrary large constant. We can find a sequence of primes $(Y <)\pi_o < \pi_1 < \dots < \pi_k$ with the following properties:

$$\pi_j + 1 = B_j \pi_{j-1}^2 T_j \quad (j = 1, \dots, k), \tag{3.1}$$

T_j is square-free, $p(T_j) > \pi_{j-1}$. The existence of such a sequence follows from the prime number theorem for arithmetical progressions and from simple sieve results.

Let $n = A_k \cdot \pi_k^2$. Then $f(n) = f(A_k)(\pi_k + 1)\pi_k = A_{k-1}\pi_{k-1}^2 T_k \pi_k$,

$$f_2(n) = f(A_{k-1})B_{k-1}\pi_{k-1}\pi_k T_{k-1} T_k \pi_{k-2}^2 = A_{k-2}\pi_{k-2}^2(\pi_{k-1}\pi_k T_{k-1} T_k).$$

Continuing, we obtain that for every $j \le k$:

$$f_j(n) = A_{k-j}\pi_{k-j}^2(\pi_{k-j+1} \dots \pi_k T_{k-j+1} \dots T_k).$$

Observe that the product in the bracket is square-free, and it is coprime to $A_{k-j}\pi_{k-j}^2$. We obtain that

$$\frac{f(n)}{n} = \frac{f(A_k)}{A_k}(1 + \frac{1}{\pi_k}),$$

$$\frac{f_2(n)}{f_1(n)} = \frac{f(A_{k-1})}{A_{k-1}}(1 + \frac{1}{\pi_{k-1}}), \dots, \frac{f_k(n)}{f_{k-1}(n)} = \frac{f(A_1)}{A_1}(1 + \frac{1}{\pi_o}).$$

Since π_o can be chosen to be arbitrarily large, it is enough to prove that there exists such a choice of $B_1, \dots, B_k, A_k$ for which

$$\frac{f(A_j)}{A_j} \in (\alpha_j, \alpha_j + \frac{\varepsilon}{2}) \quad (j = 1, \dots, k). \tag{3.2}$$

Since $\sum_{p \in \mathcal{P}} 1/p = \infty$, therefore we can choose finitely many primes, $q_1 < q_2 < \dots < q_r$ for which

$$\alpha_1 < \prod_{j=1}^{r}(1 + \frac{1}{q_j}) < \alpha_1 + \frac{\varepsilon}{2}.$$

Let $A_k = q_1^2 \dots q_r^2$. Then

$$\frac{f(A_k)}{A_k} = \prod_{j=1}^{r}(1+\frac{1}{q_j}),$$

thus (3.2) is satisfied for $j = k$. Let T be a large positive integer. We shall write B_k as $U_k V_k$, where

$$U_k = 2^T \prod_{p|f(A_k)} p^T, \quad V_k = (p_1 p_2 \dots p_s)^2,$$

where $p_1 < p_2 < \dots < p_s$ is such a collection of primes for which $p_1 >$ $> P(f(A_k))$, and

$$\alpha_2 < \prod_{j=1}^{s}(1+1/p_j) < \alpha_2 + \frac{\varepsilon}{4}.$$

We can write $A_{k-1} = (f(A_k)U_k)V_k$. We have:

$$(f(A_k)U_k, V_k) = 1.$$

Since for $r \geq 3$, $1 \leq \frac{f(p^r)}{p^r} < 1 + \frac{c}{p^{r/2}}$, therefore

$$1 \leq \frac{f(f(A_k)U_k)}{f(A_k)U_k} \leq (1+\frac{1}{2^{T/2}}) \prod_{p|f(A_k)} (1+\frac{1}{p^{T/2}}) \leq \prod_{p\in\mathcal{P}}(1+\frac{1}{p^{T/2}}).$$

The right hand side as a function of T is $1+o(1)$ for $T \to \infty$. Thus, with a suitable large T we obtain (3.2) for $j = 2$. Repeating this argument, we get (3.2) for every j. ◊

Remarks. 1.) From Th. 1 we get immediately that

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x | \frac{f_k(n)}{n} < \alpha\} = G(\alpha)$$

exists and $G(\alpha)$ *is strictly monotonic in* $(1, \infty)$.

2.) By using our method we can prove that *for every* $\alpha > 1$, $\beta > 1$,

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x | \frac{f_k(n)}{n} \in (\alpha, \alpha+\epsilon), \ \frac{f_k(n+1)}{n+1} \in (\beta, \beta+\epsilon)\}$$

exists and it is positive for $\epsilon > 0$.

References

[1] ERDŐS, P.: Some remarks on the iterates of the σ and φ functions, *Colloq. Math.* **17** (1967), 195–202.

[2] ERDŐS, P.: Private communication dated October 28, 1973 to M. V. Subbarao.

[3] ERDŐS, P. and SUBBARAO, M.V.: On the iterates of some arithmetic functions, The theory of Arithmetic Functions, Lecture Notes #251, Springer-Verlag, New York, 1972.

[4] KÁTAI, I. and WIJSMULLER, M.: On the iterates of sums of unitary divisors, *Acta Math. Hung.* **79** (1998), 149–167.
[5] LUCHT, L.: On the sum of exponential divisors and its iterates, *Archiv der Math.* **27** (1976), 283–286.
[6] STRAUS, G. and SUBBARAO, M.V.: On exponential divisors, *Duke Math. J.* **41** (1974), 465–471.
[7] SUBBARAO, M.V.: On some arithmetical convolutions, The theory of arithmetic functions, *Lecture Notes* **251**, Springer-Verlag, New York, 1972.