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WEAK COMPACTNESS AND CONVERGENCE IN TWO-PARAMETER MARTINGALE HARDY SPACES AND THE DUAL OF THE *VMO* SPACE

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Abstract: A new characterization of the two-parameter BMO spaces is given. It is proved that a set A from the two-parameter Hardy space H_1 is weakly sequentially compact if and only if the set of the maximal functions of A is uniformly integrable. We obtain that H_1 is weakly complete. Using these facts, we give a new proof for the duality between VMO and H_1.

1. Introduction

It is known that a set from L_1 is weakly sequentially compact if and only if the set is uniformly integrable (see Dunford and Schwartz [6, p. 294]). Moreover, L_1 is weakly complete (see [6, p. 290]).

The corresponding results for one-parameter Hardy spaces can be found in Dellacherie, Meyer and Yor [5] and Long [8]. Every function from the one-parameter BMO space can be written as a sum of bounded

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functions (see Garsia [7]). With the help of this characterization Long [8] proved that a set $A \subset H_1$ is weakly sequentially compact if and only if the set of the maximal functions of A is uniformly integrable. Moreover, he proved that H_1 is weakly complete.

Coifman and Weiss [4] proved for the classical Hardy spaces that the dual of VMO is H_1. This result for one-parameter martingale spaces is due to Schipp [11]. The author [13, 15] generalized this for two-parameter martingale spaces.

In this paper we generalize these results for two-parameter Hardy and BMO spaces. First we give a new characterization of the two-parameter BMO spaces, we decompose every BMO function into a sum of infinitely many bounded functions. The bounded linear functionals of H_1 are written in a new, closed form. Then we verify that $A \subset H_1$ is weakly sequentially compact if and only if the set of the maximal functions of A is uniformly integrable. We derive that H_1 is weakly complete, i.e. if $E(f^n g)$ has a limit for all $g \in BMO$, where $f^n \in H_1$, then there exists $f \in H_1$ such that f^n converges weakly to f. We prove also a weaker form of this result: if $E(f^n g)$ has a limit for all $g \in VMO$ then there exists $f \in H_1$ such that f^n converges weakly to f.

In the last section we suppose that every σ-algebra is generated by finitely many atoms. We give a new, functional analytic proof for the fact that the dual of VMO is H_1.

2. Preliminaries and notations

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_{n,m}; n, m \in \mathbb{N})$ be a non-decreasing sequence of σ-algebras with respect to the partial ordering on $\mathbb{N}^2$. Moreover, let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{F}$, i.e. $\mathcal{A} = \sigma(\mathcal{F})$. Introduce the following σ-algebras:

$$\mathcal{F}_{n,\infty} := \sigma(\cup_{k=0}^{\infty} \mathcal{F}_{n,k}), \qquad \mathcal{F}_{\infty,m} := \sigma(\cup_{k=0}^{\infty} \mathcal{F}_{k,m}).$$

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_{n,m}$, $\mathcal{F}_{n,\infty}$ and $\mathcal{F}_{\infty,m}$ $(n, m \in \mathbb{N})$ are denoted by E, $E_{n,m}$, $E_{n,\infty}$ and $E_{\infty,m}$, respectively. We briefly write L_p instead of the real or complex $L_p(\Omega, \mathcal{A}, P)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$ $(0 < p \leq \infty)$. For simplicity, we assume that for a function $f \in L_1$ we have $E_{n,0} f = E_{0,n} f = 0$ $(n \in \mathbb{N})$. The space L will denote the step functions.

We suppose that the condition (F_4) introduced by Cairoli and Walsh [3] is satisfied: for arbitrary $n, m \in \mathbb{N}$ the σ-algebras $\mathcal{F}_{n,\infty}$ and $\mathcal{F}_{\infty,m}$ are conditionally independent relative to $\mathcal{F}_{n,m}$ or, equivalently,

$(F_4) \qquad E_{n,m}f = E_{n,\infty}(E_{\infty,m}f) = E_{\infty,m}(E_{n,\infty}f) \qquad (n, m \in \mathbb{N})$

for all bounded functions f.

An integrable sequence $f = (f_{n,m}; n, m \in \mathbb{N})$ is said to be a *martingale* if

(i) $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable for all $n, m \in \mathbb{N}$

(ii) $E_{k,l}f_{n,m} = f_{k,l}$ for all $k \leq n$ and $l \leq m$.

For simplicity, we always suppose that for a martingale f we have $f_{n,0} = f_{0,n} = 0$ $(n \in \mathbb{N})$. Of course, the theorems that are to be proved later are true with a slightly modification without this condition, too.

In this paper we suppose that the stochastic basis $\mathcal{F}$ is *regular*, i.e. there exists a number $R > 0$ such that $f_{n,m} \leq Rf_{n-1,m}$, $f_{n,m} \leq$ $\leq Rf_{n,m-1}$ $(n, m \in \mathbb{N})$ hold for all non-negative martingales $(f_{n,m}; n, m \in$ $\in \mathbb{N})$.

The easiest example for a regular $\mathcal{F}$ is the sequence of dyadic σ-algebras where $\Omega = [0,1) \times [0,1)$, $\mathcal{A}$ is the σ-algebra of Borel sets, P is Lebesgue measure and

$$\mathcal{F}_{n,m} := \sigma\Big\{ \Big[\frac{k}{2^n}, \frac{k+1}{2^n}\Big) \times \Big[\frac{l}{2^m}, \frac{l+1}{2^m}\Big) : 0 \leq k < 2^n, 0 \leq l < 2^m \Big\}.$$

The stochastic basis generated by bounded Vilenkin systems are also regular (cf. Weisz [15].

We say that a sequence $X = (X_{n,m}; n, m \in \mathbb{N})$ is *uniformly integrable* if

$$\lim_{y\to\infty} \sup_{n,m\in\mathbb{N}} \int_{\{|f_{n,m}|>y\}} |f_{n,m}|\, dP = 0.$$

The *maximal function* of a sequence $X = (X_{n,m}; n, m \in \mathbb{N})$ is denoted by

$$X^* := \sup_{n,m\in\mathbb{N}} |X_{n,m}|.$$

The *martingale Hardy space* H_p $(0 < p \leq \infty)$ denotes the space of all martingales $f = (f_{n,m}; n, m \in \mathbb{N})$ for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

It is proved in Brossard [1, 2] and Weisz [15] that, for all $0 < p < \infty$, H_p is equivalent to the space generated by the *quadratic variation*, i.e. by

$$S(f) := \Big(\sum_{n,m\in\mathbb{N}} |f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}|^2 \Big)^{1/2}.$$

Moreover, it is known that $H_p \sim L_p$ for all $1 < p < \infty$, where $\sim$ denotes the equivalence of the spaces and norms.

The dual of H_1 is the BMO space (see Weisz [13, 15]) where BMO denotes the space of those functions $f \in L_2$ for which

$$\|f\|_{BMO} = \sup_{\nu} P(\nu \neq \infty)^{-1/2} \|f - f^{\nu}\|_2 < \infty.$$

Here the supremum is taken over all two-parameter stopping times and f^{ν} denotes the stopped martingale.

For an arbitrary space $\mathbf{Z}$ we denote by $\mathbf{Z}(\mathcal{F}_{n,m})$ the $\mathcal{F}_{n,m}$ measurable functions from $\mathbf{Z}$.

3. A characterization of BMO

In this section we characterize the dual of H_1. Let $\theta = (\theta_{n,m}; n, m \in \in \mathbb{N})$ be a sequence of functions such that there exist N and M for which

$$\begin{aligned} \theta_{n,m} &= \theta_{N,m} \qquad \text{for all } n \geq N,\ m \leq M-1 \\ \theta_{n,m} &= \theta_{n,M} \qquad \text{for all } m \geq M,\ n \leq N-1 \\ \theta_{n,m} &= \theta_{N,M} \qquad \text{for all } n \geq N,\ m \geq M. \end{aligned}$$

We say that a sequence θ of this type is in the space $\mathcal{H}$ if

$$\|\theta\|_{\mathcal{H}} := \|\theta^*\|_1 < \infty.$$

Now we give the dual of $\mathcal{H}$.

Lemma 1. *For all bounded linear functionals l of $\mathcal{H}$ there exist sequences $(\epsilon_{n,m}; n, m \in \mathbb{N})$, $(\xi_{n,\infty}; n \in \mathbb{N})$, $(\xi_{\infty,m}; m \in \mathbb{N})$ of functions and a function ψ such that*

$$\|l\| \sim \Big\| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\epsilon_{n,m}| + \sum_{n=0}^{\infty} |\xi_{n,\infty}| + \sum_{m=0}^{\infty} |\xi_{\infty,m}| + |\psi| \Big\|_{\infty}, \tag{1}$$

$$\begin{aligned} l(\theta) = {} & \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} E(\theta_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{N-1} E\Big[\theta_{n,M}(\xi_{n,\infty} + \sum_{m=M}^{\infty} \epsilon_{n,m})\Big] + \\ & + \sum_{m=0}^{M-1} E\Big[\theta_{N,m}(\xi_{\infty,m} + \sum_{n=N}^{\infty} \epsilon_{n,m})\Big] + \\ & + E\Big[\theta_{N,M}\Big(\sum_{n=N}^{\infty} \sum_{m=M}^{\infty} \epsilon_{n,m} + \sum_{n=N}^{\infty} \xi_{n,\infty} + \sum_{m=M}^{\infty} \xi_{\infty,m} + \psi\Big)\Big]. \end{aligned} \tag{2}$$

Proof. It is easy to see that if l is of the form (2) then l is a bounded linear functional on $\mathcal{H}$ and $\|l\|$ can be estimated by the right hand side of (1).

Let l be an arbitrary bounded linear functional on $\mathcal{H}$. It follows from the duality between L_1 and L_∞ and from the linearity of l, that there exist $\epsilon_{n,m}$, $\xi_{n,m}$, $\psi_{n,m} \in L_\infty$ $(n, m \in \mathbb{N})$ such that

$$l(\theta) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} E(\theta_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{N-1} E(\theta_{n,M}\xi_{n,M}) + \tag{3}$$
$$+ \sum_{m=0}^{M-1} E(\theta_{N,m}\xi_{N,m}) + E(\theta_{N,M}\psi_{N,M}).$$

Writing $\theta_{n,m} := \text{sign}\ \epsilon_{n,m}$ $(n \le N-1, m \le M-1)$, $\theta_{n,M} := \text{sign}\ \xi_{n,M}$ $(n \le N-1)$, $\theta_{N,m} := \text{sign}\ \xi_{N,m}$ $(m \le M-1)$ and $\theta_{N,M} := \text{sign}\ \psi_{N,M}$, we can see that

$$\left\| \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\epsilon_{n,m}| + \sum_{n=0}^{N-1} |\xi_{n,M}| + \sum_{m=0}^{M-1} |\xi_{N,m}| + |\psi_{N,M}| \right\|_\infty \le \|l\| \tag{4}$$

for all $N, M \in \mathbb{N}$. If $N = 0$ then we have

$$\left\| \sum_{m=0}^{M-1} |\xi_{0,m}| + |\psi_{0,M}| \right\|_\infty \le \|l\|. \tag{5}$$

The bounded linear functional l from (3) can be rewritten in

(6)

$$l(\theta) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} E(\theta_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{N-1} E(\theta_{n,M}\xi_{n,L}) + \sum_{n=0}^{N-1}\sum_{m=M}^{L-1} E(\theta_{n,M}\epsilon_{n,m}) +$$
$$+ \sum_{m=0}^{M-1} E(\theta_{N,m}\xi_{K,m}) + \sum_{n=N}^{K-1}\sum_{m=0}^{M-1} E(\theta_{N,m}\epsilon_{n,m}) + \sum_{n=N}^{K-1}\sum_{m=M}^{L-1} E(\theta_{N,M}\epsilon_{n,m}) +$$
$$+ \sum_{n=N}^{K-1} E(\theta_{N,M}\xi_{n,L}) + \sum_{m=M}^{L-1} E(\theta_{N,M}\xi_{K,m}) + E(\theta_{N,M}\psi_{K,L})$$

where $K \ge N$ and $L \ge M$. (3) and (6) imply

$$\xi_{n,M} = \xi_{n,L} + \sum_{m=M}^{L-1} \epsilon_{n,m}, \tag{7}$$

$$\xi_{N,m} = \xi_{K,m} + \sum_{n=N}^{K-1} \epsilon_{n,m}, \tag{8}$$

$$\psi_{N,M} = \psi_{K,L} + \sum_{n=N}^{K-1}\sum_{m=M}^{L-1} \epsilon_{n,m} + \sum_{n=N}^{K-1} \xi_{n,L} + \sum_{m=M}^{L-1} \xi_{K,m}. \tag{9}$$

We can conclude from (4) and (7) that there exists the limit $\xi_{n,\infty} : := \lim_{L\to\infty} \xi_{n,L}$ and

$$|\xi_{n,\infty} - \xi_{n,M}| \le \sum_{m=M}^{\infty} |\epsilon_{n,m}|. \tag{10}$$

Similarly,

$$|\xi_{\infty,m} - \xi_{N,m}| \le \sum_{n=N}^{\infty} |\epsilon_{n,m}|. \tag{11}$$

Since N and M are arbitrary, (10) and (11) hold for all $n, m \in \mathbb{N}$.

We prove now that

$$\lim_{K,L\to\infty} \sum_{n=N}^{K-1} \xi_{n,L} = \sum_{n=N}^{\infty} \xi_{n,\infty}. \tag{12}$$

Indeed, (4), (5) and (10) imply

$$\begin{aligned} |\sum_{n=N}^{\infty} \xi_{n,\infty} - \sum_{n=N}^{K-1} \xi_{n,L}| &\le \sum_{n=N}^{K-1} |\xi_{n,\infty} - \xi_{n,L}| + \sum_{n=K}^{\infty} |\xi_{n,\infty}| \le \\ &\le \sum_{n=N}^{K-1} \sum_{m=L}^{\infty} |\epsilon_{n,m}| + \sum_{n=K}^{\infty} |\xi_{n,\infty} - \xi_{n,0}| + \sum_{n=K}^{\infty} |\xi_{n,0}| \le \\ &\le \sum_{n=0}^{\infty} \sum_{m=L}^{\infty} |\epsilon_{n,m}| + \sum_{n=K}^{\infty} \sum_{m=0}^{\infty} |\epsilon_{n,m}| + \sum_{n=K}^{\infty} |\xi_{n,0}| \le \delta \end{aligned}$$

if K and L are large enough. Similarly,

$$\lim_{K,L\to\infty} \sum_{m=M}^{L-1} \xi_{K,m} = \sum_{m=M}^{\infty} \xi_{\infty,m}. \tag{13}$$

Substituting (12) and (13) in (9) we get that $\lim_{K,L\to\infty} \psi_{K,L} = \psi$ does exist and

$$\psi_{N,M} = \psi + \sum_{n=N}^{\infty} \sum_{m=M}^{\infty} \epsilon_{n,m} + \sum_{n=N}^{\infty} \xi_{n,\infty} + \sum_{m=M}^{\infty} \xi_{\infty,m}.$$

This together with (3), (7) and (8) prove (2). In the same way we can verify that

$$\lim_{K,L\to\infty} \sum_{n=N}^{K-1} |\xi_{n,L}| = \sum_{n=N}^{\infty} |\xi_{n,\infty}|, \qquad \lim_{K,L\to\infty} \sum_{m=M}^{L-1} |\xi_{K,m}| = \sum_{m=M}^{\infty} |\xi_{\infty,m}|,$$

which finishes the proof of (1). $\Diamond$

Now we formulate the main result of this section. Its one-parameter version can be found in Garsia [7] and Long [8].

Theorem 1. *A function g is in BMO if and only if there exist functions $\epsilon_{n,m}$, $\xi_{n,\infty}$, $\xi_{\infty,m}$ and ψ $(n,m \in \mathbb{N})$ such that*

$$g=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E_{n,m}\epsilon_{n,m}+\sum_{n=0}^{\infty}E_{n,\infty}\xi_{n,\infty}+\sum_{m=0}^{\infty}E_{\infty,m}\xi_{\infty,m}+\psi, \tag{14}$$

$$\|g\|_{BMO}\sim\Big\|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\epsilon_{n,m}|+\sum_{n=0}^{\infty}|\xi_{n,\infty}|+\sum_{m=0}^{\infty}|\xi_{\infty,m}|+|\psi|\Big\|_{\infty}. \tag{15}$$

Proof. It is known that for all $g \in BMO$ there exists a bounded linear functional l on H_1 such that $\|l\| \sim \|g\|_{BMO}$ and

$$l_g(f)=l(f)=E(fg) \qquad (f\in L_2), \tag{16}$$

where L_2 is dense in H_1 (see Weisz [15]). We shall use also the notation $\langle f,g\rangle := l_g(f)$ $(f \in H_1)$. The subspace $\cup_{n,m\in\mathbb{N}}L_\infty(\mathcal{F}_{n,m})$, which is dense in H_1, can be embedded isometrically into $\mathcal{H}$. If $g \in BMO$ then the bounded linear functional l given in (16) can be extended onto $\mathcal{H}$ norm preserving. By Lemma 1 then there exist $\epsilon_{n,m}$, $\xi_{n,\infty}$, $\xi_{\infty,m}$ and ψ $(n,m \in \in \mathbb{N})$ such that (1) and (2) hold. Since $\|l\| \sim \|g\|_{BMO}$, (15) is proved. By (2),

$$\begin{aligned}E(f_{N,M}g)=l(\theta)&=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}E(f_{n,m}\epsilon_{n,m})+\\
&+\sum_{n=0}^{N-1}E\Big[f_{n,M}(\xi_{n,\infty}+\sum_{m=M}^{\infty}\epsilon_{n,m})\Big]+\sum_{m=0}^{M-1}E\Big[f_{N,m}(\xi_{\infty,m}+\sum_{n=N}^{\infty}\epsilon_{n,m})\Big]+\\
&+E\Big[f_{N,M}\Big(\sum_{n=N}^{\infty}\sum_{m=M}^{\infty}\epsilon_{n,m}+\sum_{n=N}^{\infty}\xi_{n,\infty}+\sum_{m=M}^{\infty}\xi_{\infty,m}+\psi\Big)\Big]=\\
&=E\Big[f_{N,M}\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E_{n,m}\epsilon_{n,m})+\sum_{n=0}^{\infty}E_{n,\infty}\xi_{n,\infty}+\sum_{m=0}^{\infty}E_{\infty,m}\xi_{\infty,m}+\psi\Big)\Big].\end{aligned}$$

Since $f_{N,M}$ is arbitrary, (14) is proved.

On the other hand, if g can be written in (14) then $\epsilon_{n,m}$, $\xi_{n,\infty}$, $\xi_{\infty,m}$ and ψ $(n,m \in \mathbb{N})$ give a bounded linear functional on $\mathcal{H}$ and hence on the image of $\cup_{n,m\in\mathbb{N}}L_\infty(\mathcal{F}_{n,m})$. By the density this yields that $E(f_{N,M}g)$ can be extended to a bounded linear functional on H_1. Consequently, $g \in BMO$. The proof of Th. 1 is complete. ◊

Theorem 2. *If $g \in BMO$ then there exist $\epsilon_{n,m}$, $\xi_{n,\infty}$, $\xi_{\infty,m}$ and ψ $(n,m \in \mathbb{N})$ such that (15) hold and the corresponding bounded linear functional can be written in the form*

$$
\begin{aligned}
\langle f, g\rangle = &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(f_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{\infty} E(f_{n,\infty}\xi_{n,\infty}) + \\
&+ \sum_{m=0}^{\infty} E(f_{\infty,m}\xi_{\infty,m}) + E(f\psi)
\end{aligned} \tag{17}
$$

where $f \in H_1$.

Proof. We apply Th. 1. By (15) the series (14) converges also in L_1 norm. Hence, for $f \in L_\infty$ we have

$$
\begin{aligned}
\langle f, g\rangle = E(fg) = &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(fE_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{\infty} E(fE_{n,\infty}\xi_{n,\infty}) + \\
&+ \sum_{m=0}^{\infty} E(fE_{\infty,m}\xi_{\infty,m}) + E(f\psi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(f_{n,m}\epsilon_{n,m}) + \\
&+ \sum_{n=0}^{\infty} E(f_{n,\infty}\xi_{n,\infty}) + \sum_{m=0}^{\infty} E(f_{\infty,m}\xi_{\infty,m}) + E(f\psi).
\end{aligned}
$$

In case $f \in H_1$ is arbitrary, we choose $f^k \in L_\infty$ such that $f^k \to f$ in H_1 norm. Then

$$
\begin{aligned}
\langle f, g\rangle = \lim_{k\to\infty} E(f^k g) = \lim_{k\to\infty} \Big[&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(f^k_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{\infty} E(f^k_{n,\infty}\xi_{n,\infty}) + \\
&+ \sum_{m=0}^{\infty} E(f^k_{\infty,m}\xi_{\infty,m}) + E(f^k\psi)\Big].
\end{aligned}
$$

Since $|f^k_{n,m} - f_{n,m}| \le (f^k - f)^*$ $(n, m \in \mathbb{N} \cup \{\infty\})$, (15) completes the proof of (17). ◊

4. Weak compactness and convergence in $\mathbf{H_1}$

We denote the weak topology in H_1 by $\sigma(H_1, BMO)$ and the weak* topology by $\sigma(BMO, H_1)$. The following lemma can be found in Long [8, p. 61].

Lemma 2. *Let A be a set of sequences $X = (X_{n,m})$ such that*

$$\sup_{X\in A} E(X^*) \le C.$$

Suppose that for all measurable functions T from Ω to $\mathbb{N}^2$, $A_T := \{X_T : : X \in A\}$ is uniformly integrable, where $X_T := \sum_{n,m\in\mathbb{N}} X_{n,m} 1_{\{T=(n,m)\}}$. Then $A^* := \{X^* : X \in A\}$ is uniformly integrable, too.

Now we show that under some conditions the sum in (17) is uniformly.

Lemma 3. *Let $g \in BMO$, $A \subset H_1$ and A^* be uniformly integrable. Then for all $\delta > 0$ there exist N_0 and M_0 such that for $N \geq N_0$ and $M \geq M_0$,*

$$\sup_{f\in A}\Big(\Big|\sum_{\substack{n\geq N \text{ or}\\ m\geq M}} E(f_{n,m}\epsilon_{n,m})\Big| + \Big|\sum_{n=N}^{\infty} E(f_{n,\infty}\xi_{n,\infty})\Big| + $$
$$+\Big|\sum_{m=M}^{\infty} E(f_{\infty,m}\xi_{\infty,m})\Big|\Big) \leq \delta,$$

where $\epsilon_{n,m}$, $\xi_{n,\infty}$ and $\xi_{\infty,m}$ appear in Th. 2.

Proof. We prove the lemma for the first sum, only. Since

$$\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}|\epsilon_{n,m}|$$

converges to $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\epsilon_{n,m}|$ also in measure, for all $\eta > 0$ there exist N_0 and M_0 such that for $N \geq N_0$ and $M \geq M_0$ we have

$$F_{N,M} := \Big|\Big\{\omega : \sum_{\substack{n\geq N \text{ or}\\ m\geq M}} |\epsilon_{n,m}| > \delta/(6C)\Big\}\Big| \leq \eta$$

where

$$C \geq \sup\Big(\sup_{f\in A} E(f^*), \Big\|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\epsilon_{n,m}| + \sum_{n=0}^{\infty}|\xi_{n,\infty}| + \sum_{m=0}^{\infty}|\xi_{\infty,m}| + |\psi|\Big\|_{\infty}\Big).$$

By the uniform integrability of A^*, η can be chosen such that

$$\sup_{f\in A}\int_F f^*\, dP \leq \delta/(6C) \quad \text{if} \quad P(F) \leq \eta.$$

Thus we obtain for all $f \in A$ that

$$\Big|\sum_{\substack{n\geq N \text{ or}\\ m\geq M}} E(f_{n,m}\epsilon_{n,m})\Big| \leq C\int_{F_{N,M}} f^*\, dP + \delta/(6C)\int_{\Omega\setminus F_{N,M}} f^*\, dP \leq \delta/3,$$

which finishes the proof of Lemma 3. $\diamondsuit$

Lemma 4. *If f^k is a weak Cauchy sequence of H_1 then for all measurable $T : \Omega \mapsto \mathbb{N}^2$, f_T^k is weakly convergent in L_1.*

Proof. For all $h \in L_\infty$,

$$E(f_T^k h) = E\Big(\sum_{n,m\in\mathbb{N}} f_{n,m}^k h 1_{\{T=(n,m)\}}\Big) = E\Big(\sum_{n,m\in\mathbb{N}} f_{n,m}^k \epsilon_{n,m}\Big) = \langle f^k, g\rangle$$

where $\epsilon_{n,m} := h1_{\{T=(n,m)\}}$ and $g := \sum_{n,m\in\mathbb{N}} E_{n,m}\epsilon_{n,m} \in BMO$ by Th. 1. Since $\langle f^k, g\rangle$ converges, we conclude that f_T^k is a weak Cauchy sequence in L_1. The weak completeness of L_1 (see Dunford, Schwartz [6, p. 290]) shows the lemma. $\Diamond$

Theorem 3. *A set $A \subset H_1$ is weakly sequentially compact if and only if A^* is uniformly integrable.*

Proof. The uniform integrability of A^* implies that A is also uniformly integrable. This means that $A \subset L_1$ is weakly sequentially compact in the topology $\sigma(L_1, L_\infty)$ (see Dunford and Schwartz [6, p. 294]). Thus for all infinite sequence from A there exist a subsequence $(f^k) \subset A$ and a function $f \in L_1$ such that $f^k \to f$ in $\sigma(L_1, L_\infty)$, i.e. $\lim_{k\to\infty} E(f^k h) = E(fh)$ for all $h \in L_\infty$. It is easy to see that also $\lim_{k\to\infty} E(f^k_{n,m} h) = E(f_{n,m} h)$ for all $n, m \in \mathbb{N}\cup\{\infty\}$ and $h \in L_\infty$. For an arbitrary $g \in BMO$, Th. 2 and Lemma 3 imply

$$\lim_{k\to\infty}\langle f^k, g\rangle = \lim_{k\to\infty}\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(f^k_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{\infty} E(f^k_{n,\infty}\xi_{n,\infty}) + \sum_{m=0}^{\infty} E(f^k_{\infty,m}\xi_{\infty,m}) + E(f^k\psi)\Big) =$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lim_{k\to\infty} E(f^k_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{\infty} \lim_{k\to\infty} E(f^k_{n,\infty}\xi_{n,\infty}) + \sum_{m=0}^{\infty} \lim_{k\to\infty} E(f^k_{\infty,m}\xi_{\infty,m}) + E(f\psi) +$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(f_{n,m}\epsilon_{n,m}) + \sum_{n=0}^{\infty} E(f_{n,\infty}\xi_{n,\infty}) = + \sum_{m=0}^{\infty} E(f_{\infty,m}\xi_{\infty,m}) + E(f\psi) = \langle f, g\rangle.$$

We only have to show that $f \in H_1$. For any measurable $T : \Omega \mapsto \mathbb{N}^2$ the last equation implies that

$$E(|f_T|) = \sum_{n,m\in\mathbb{N}} E(f_{n,m}\epsilon_{n,m}) \le \sup_{k\in\mathbb{N}}\Big|\sum_{n,m\in\mathbb{N}} E(f^k_{n,m}\epsilon_{n,m})\Big| \le$$

$$\le \sup_{k\in\mathbb{N}} \sum_{n,m\in\mathbb{N}} E((f^k)^*|\epsilon_{n,m}|) \le C,$$

where $\epsilon_{n,m} := \text{sign } f_{n,m}1_{\{T=(n,m)\}}$. Taking the supremum over all T's we obtain $f \in H_1$.

Now assume that A is weakly sequentially compact in $\sigma(H_1, BMO)$. If a subsequence $(f^k) \subset H_1$ converges weakly to f, then, by Lemma 4,

$$E(f_T^k h) = \langle f^k, g \rangle \longrightarrow \langle f, g \rangle = E(f_T h),$$

where $h \in L_\infty$ is arbitrary. Consequently, A_T is weakly sequentially compact in $\sigma(L_1, L_\infty)$, and so A_T is uniformly integrable. Since A is bounded in H_1, Lemma 2 shows the uniform integrability of A^*. The proof of the theorem is complete. $\Diamond$

Now we can verify the weak completeness of H_1.

Theorem 4. *The space H_1 is weakly complete, i.e. if f^k is a sequence from H_1 such that $\langle f^k, g \rangle$ has a limit for all $g \in BMO$, then there exists $f \in H_1$ such that $\langle f^k, g \rangle \to \langle f, g \rangle$.*

Proof. Every f^k defines a bounded linear functional on BMO with equivalent norm $\|f^k\|_{H_1}$. By the convergence, these functionals are pointwise bounded. The Banach-Steinhaus theorem implies that they are uniformly bounded, thus $A := \{f^k\}$ is a bounded set in H_1. It follows from Lemma 4 that A_T is weakly sequentially compact in $\sigma(L_1, L_\infty)$. Then A_T is uniformly integrable and hence A^* is also uniformly integrable (see Lemma 2). Th. 3 yields the weak sequentially compactness of A in $\sigma(H_1, BMO)$. Thus there is a subsequence (f^{k_n}) and $f \in H_1$ such that $\langle f^{k_n}, g \rangle \to \langle f, g \rangle$ for all $g \in BMO$. The proof can be finished as usual. $\Diamond$

We formulate a relation between weak and strong convergence in H_1.

Theorem 5. *Suppose that the sequence (f^k) from H_1 converges weakly to $f \in H_1$. Then the convergence holds in H_1 norm if and only if*

$$\|\sup_{n\in\mathbb{N}} |E_{n,\infty}(f - f^k)|\|_1 \to 0.$$

Proof. The "only if" part is clear. To prove the other part observe that $A := \{f^k\}$ is weakly sequentially compact in H_1, hence A^* is uniformly integrable (see Th. 2). The convergence $f^k \to f$ in H_1 norm follows from the uniform integrability of A^* and from the inequality

$$P\big((f - f^k)^* > \delta\big) \le \frac{1}{\delta} \|\sup_{n\in\mathbb{N}} |E_{n,\infty}(f - f^k)|\|_1.$$

Note that this inequality was proved by the author in [15, p. 85]. $\Diamond$

Now we verify a weaker version of Th. 4.

Theorem 6. *Let $\mathbf{Z}$ be the closure of $\cup_{n,m\in\mathbb{N}} BMO(\mathcal{F}_{n,m})$ in BMO norm. If f^k is a sequence from H_1 such that $\langle f^k, g \rangle$ has a limit for all $g \in \mathbf{Z}$, then there exists $f \in H_1$ such that $\langle f^k, g \rangle \to \langle f, g \rangle$.*

Proof. Every f^k defines a bounded linear functional on $\mathbf{Z}$. Since

$$\|f_{n,m}\|_{H_1} \sim \sup_{\substack{g\in BMO(\mathcal{F}_{n,m})\\ \|g\|_{BMO}\le 1}} |\langle f_{n,m}, g\rangle| \tag{18}$$

for all $n,m \in \mathbb{N}$ and $\sup_{n,m\in\mathbb{N}} \|f_{n,m}\|_{H_1} = \|f\|_{H_1}$, we have $A := \{f^k\}$ is again a bounded set in H_1. If $\langle f^k, g\rangle$ converges for all $g \in \mathbf{Z}$ then it converges also for all $g \in BMO(\mathcal{F}_{n,m})$. By Th. 4 there exists a function $f_{n,m} \in H_1$ $(n,m \in \mathbb{N})$ such that

$$\langle f^k, g\rangle = \langle f^k_{n,m}, g\rangle \longrightarrow \langle f_{n,m}, g\rangle \qquad (g \in BMO(\mathcal{F}_{n,m})). \tag{19}$$

Then $\langle f^k, g\rangle = \langle f^k_{n+1,m}, g\rangle \longrightarrow \langle f_{n+1,m}, g\rangle$ $(g \in BMO(\mathcal{F}_{n+1,m}))$. Since $\langle f^k_{n,m}, g\rangle = \langle f^k_{n+1,m}, g\rangle$ for all $g \in BMO(\mathcal{F}_{n,m})$, we can see that $f_{n,m} = E_{n,m} f_{n+1,m}$. Consequently, $f := (f_{n,m})$ is a martingale. (18) and (19) imply that

$$\|f_{n,m}\|_{H_1} \le C \sup_{k\in\mathbb{N}} \|f^k_{n,m}\|_{H_1} \le C \sup_{k\in\mathbb{N}} \|f^k\|_{H_1} \le C. \tag{20}$$

Thus $f \in H_1$. For $g \in \mathbf{Z}$ we can choose $g_0 \in \cup_{n,m\in\mathbb{N}} BMO(\mathcal{F}_{n,m})$ such that $\|g - g_0\|_{BMO} \le \delta$. Then

$$|\langle f, g\rangle - \langle f^k, g\rangle| \le |\langle f, g\rangle - \langle f, g_0\rangle| + |\langle f, g_0\rangle - \langle f^k, g_0\rangle| + \\ + |\langle f^k, g_0\rangle - \langle f^k, g\rangle| \le \epsilon$$

if k is large enough. The proof of the theorem is complete. $\Diamond$

5. The dual of VMO

In this section we suppose that every σ-algebra $\mathcal{F}_{n,m}$ is generated by finitely many atoms. Let VMO denote the closure of $\cup_{n,m\in\mathbb{N}} L(\mathcal{F}_{n,m})$ in the BMO norm. Recall that $L(\mathcal{F}_{n,m})$ denotes the $\mathcal{F}_{n,m}$ measurable step functions.

Theorem 7. *The dual of VMO is H_1.*

Proof. It follows from the duality between H_1 and BMO that every $g \in VMO$ defines a bounded linear functional $l_f(g) := \langle f, g\rangle$ on VMO with $\|l_f\| \le C\|f\|_{H_1}$.

To prove the other side let $l \in VMO^*$ be an arbitrary bounded linear functional. By the Banach-Alaoglu theorem

$$Y := \{k \in VMO^* : \|k\| \le \|l\|\}$$

is compact with respect to the weak*, i.e. the $\sigma(VMO^*, VMO)$ topology (see e.g. Dunford and Schwartz [6, p. 424]). Since VMO is separable, the weak* topology on Y is metrizable (cf. [6, p. 426]). The natural embedding of H_1 into VMO^* is dense in VMO^* with respect to $\sigma(BMO^*, BMO)$ (cf. [6, p. 425]). Hence $H_1 \cap Y$ is dense in Y with respect to $\sigma(VMO^*, VMO)$. From this it follows that there exists a sequence $(f^k) \subset H_1 \cap Y$ such that $f^k \to l$ in the $\sigma(VMO^*, VMO)$ topology. In other words, $\langle f^k, g\rangle \to \langle l, g\rangle$ for all $g \in VMO$. By Th. 6 there exists $f \in H_1$ such that $\langle f^k, g\rangle \to \langle f, g\rangle$ for all $g \in VMO$. Hence $l = f$. Moreover, (20) implies that $\|f\|_{H_1} \le C\|l\|$, which completes the proof. $\diamondsuit$

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