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NOTE ON MULTIPLICATIVE FUNCTIONS SATISFYING CONGRUENCE PROPERTY II.

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Abstract: We solve the congruence

$$f(An+B) \equiv Cf(n) + D \pmod{n} \qquad (n = 1, 2, \dots)$$

for complete multiplicative function f, where $A > 0$, $B > 0$, C, $D \neq 0$ are given integers.

An arithmetical function $f(n) \not\equiv 0$ is said to be *multiplicative* if $(n, m) = 1$ implies

$$f(nm) = f(n)f(m),$$

and it is called *completely multiplicative* if this equation holds for all pairs of positive integers n and m. In the following we denote by $\mathcal{M}$ and $\mathcal{M}^*$ the set of all integer-valued multiplicative and completely multiplicative

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functions, respectively. Let $\mathbb{N}$ denote the set of all positive integers. For each $k \in \mathbb{N}$ we denote by χ_k the Dirichlet character $(\bmod\, k)$.

The problem concerning the characterization of some arithmetical functions by congruence properties was studied by several authors. The first result of this type was found by M. V. Subbarao [6], namely he proved in 1966 that if $f \in \mathcal{M}$ satisfies the relation

$$f(n+m) \equiv f(m) \pmod{n} \quad \text{for all} \quad n, m \in \mathbb{N}, \tag{1}$$

then $f(n)$ is a power of n with non-negative integer exponent. A. Iványi in [1] extended this result proving that if $f \in \mathcal{M}^*$ and (1) holds for a fixed $m \in \mathbb{N}$ and for all $n \in \mathbb{N}$, then $f(n)$ has also the same form. In [5] we improved the results of Subbarao and Iványi mentioned above by proving the following

Theorem A. *Assume that M is a fixed positive integer and $f \in \mathcal{M}$. If $f(M) \neq 0$ and f satisfies the relation*

$$f(n+M) \equiv f(M) \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N},$$

then there is a non-negative integer α such that $f(n) = n^\alpha$ for all $n \in \mathbb{N}$.

In 1993 the first named author proved in [4] the following

Theorem B. *Let $A > 0, B > 0, C \neq 0$ and $N > 0$ be integers with the condition $(A, B) = 1$. If $f \in \mathcal{M}$ satisfies the relation*

$$f(An+B) \equiv C \pmod{n} \quad \textit{for all} \quad n \geq N,$$

then there are a non-negative integer α and a real-valued Dirichlet character χ_A such that

$$f(n) = \chi_A(n) n^\alpha \quad \textit{for all} \quad n \in \mathbb{N}, \ (n, A) = 1.$$

A generalization of Th. B was obtained in [2]. Another characterization of the function $f(n) = n^\alpha$ $(n \in \mathbb{N})$ by using congruence property was found by A. Iványi [1]. In 1972, he proved that if $f \in \mathcal{M}$ satisfies the relation

$$f(n+m) \equiv f(n) + f(m) \pmod{n} \quad \text{for all} \quad n, m \in \mathbb{N}, \tag{2}$$

then $f(n)$ is a power of n with positive integer exponent. It is proved in [3] that this result continues to hold even if the relation (2) is valid for all primes m instead of for all positive integers m.

Our purpose in this paper is to prove the following

Theorem. *Assume that $A > 0, B > 0, C, D \neq 0$ are fixed integers with $(A, B) = 1$ and a function $f \in \mathcal{M}^*$ satisfies the congruence*

$$f(An+B) \equiv Cf(n) + D \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N}. \tag{3}$$

Then the following assertions hold:

(I) *If $f(p) = 0$ for some prime p with $(p, A) = 1$, then $p = 2$, $C = -1$, $D = 1$, $(2, AB) = 1$ and $f(n) = \chi_2(n)$ for all $n \in \mathbb{N}$,*

(II) *If $f(n) \neq 0$ for all $n \in \mathbb{N}$, $(n, A) = 1$, then either*

$$C + D = 1 \quad \text{and} \quad f(n) = 1 \quad \text{for all} \quad n \in \mathbb{N}$$

or there are a non-negative integer α and a real-valued Dirichlet character χ_A such that

$$f(n) = \chi_A(n) n^\alpha \quad \text{for all} \quad n \in \mathbb{N},\ (n, A) = 1. \tag{4}$$

Proof. First we note from (3) and Th. B that (4) is satisfied if $C = 0$. In the following we assume that $C \neq 0$. The proof is based on Lemma 1 and 2.

Lemma 1. *Assume that the conditions of the theorem are satisfied. If there is a prime p such that $(p, A) = 1$ and $f(p) = 0$, then*

$$p = 2,\ C = -1,\ D = 1,\ (2, AB) = 1 \quad \text{and} \quad f(n) = \chi_2(n).$$

Proof of Lemma 1. Since $(p, A) = 1$, one can deduce that there is a positive integer n_0 such that $p | An_0 + B$, and so by (3)

$$0 = f[A(pn + n_0) + B] \equiv Cf(pn + n_0) + D \pmod{pn + n_0}$$

holds for all $n \in \mathbb{N}$. Let $m \equiv 1 \pmod p$ be a positive integer. Then we infer from the above relation that

$$-Df(m) \equiv Cf(m)f(pn+n_0) = Cf[m(pn+n_0)] \equiv -D \pmod{pn+n_0},$$

consequently

$$f(m) = 1 \quad \text{for all} \quad m \equiv 1 \pmod p.$$

This shows that $f(n) = \chi_p(n)$ is satisfied for all $n \in \mathbb{N}$. Here we have used the fact $f(n) = \chi_p(n) = 0$ for all $n \in \mathbb{N}$, $p|n$. It is clear that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$, consequently

$$|f(An + B) - Cf(n) - D| \leq 1 + |C| + |D| := E \quad \text{for all} \quad n \in \mathbb{N}.$$

By (3) we have

$$f(An + B) = Cf(n) + D \quad \text{for all} \quad n > E,$$

which gives

$$f(AMn + B) = Cf(M)f(n) + D \tag{5}$$

for all $n > E$ and for all $M \in \mathbb{N}$. By using induction on k, (5) shows that

$$f\Big((AM)^k n + B\left((AM)^{k-1} + \ldots + AM + 1\right)\Big) =$$
$$= (Cf(M))^k f(n) + D\left[(Cf(M))^{k-1} + \ldots + Cf(M) + 1\right]$$

is valid for all integers $k, M \in \mathbb{N}$, $n > E$. Therefore for $n = p^s > E$, the above relation shows that

$$|D|.|(Cf(M))^{k-1}+\ldots+Cf(M)+1\,|\leq 1 \quad \text{for all} \quad k\in\mathbb{N}.$$

If $(M,p)=1$, then $f(M)\neq 0$ and so the above relation implies $Cf(M)\neq$ $\neq 1$ and $Cf(M)=-1$ for all $M\in\mathbb{N}$, $(M,p)=1$. Thus $f(m)=1$ if $(m,p)=1$ and $f(m)=0$ if $p|m$, furthermore $C=-1$, $D=1$. Since

$$f(Apn+B)\equiv Cf(pn)+D=1 \pmod{n},$$

we have $(p,B)=1$. If $p>2$, then there is a positive integer $l\leq p-1$ such that $(l,p)=(Al+B,p)=1$, which with (3) implies

$$1=f[A(pn+l)+B]\equiv -f(pn+l)+1=0 \pmod{pn+l}.$$

This is imposibble. Thus we have $p \doteq 2$ and so $f(n)=\chi_2(n)$ for all $n\in\mathbb{N}$. Lemma 1 is proved. ◊

Lemma 2. *Assume that the conditions of the theorem are satisfied and $f(n)\neq 0$ for all $n\in\mathbb{N}$, $(n,A)=1$. If there is prime p such that $p|A$ and $f(p)=0$, then $n\mid f(n)$ for all $n\in\mathbb{N}$.*

Proof of Lemma 2. Assume that there is a prime p such that $f(p)=0$ with $p|A$. Then for each $M\in\mathbb{N}$, by (3) and our assumptions, we have

$$f(B)^2f(An+1)f[A(pM-1)n+1]=$$
$$=f(B)f[ABn(A(pM-1)+pM)+B]\equiv$$
$$\equiv Cf(B)^2f(p)f(n)f[\frac{A}{p}(pM-1)n+M]+Df(B)=Df(B) \pmod{n},$$

consequently

$$[Cf(B)f(n)+D][Cf(B)f(pM-1)f(n)+D]\equiv Df(B) \pmod{n}.$$

This with $n=p^s$, $s\to\infty$ shows that $D=f(B)$. Thus, we have

$$C^2D^2f(pM-1)f(n)^2+CD^2[f(pM-1)+1]f(n)\equiv 0 \pmod{n}$$

for all n, $M\in\mathbb{N}$. For each $m\in\mathbb{N}$, we also have

$$C^2D^2f(pM-1)f(n)^2f(m)^2+$$
$$+CD^2[f(pM-1)+1]f(n)f(m)\equiv 0 \pmod{n}.$$

which gives

$$C^2D^2f(pM-1)[f(m)^2-f(m)]f(n)^2\equiv 0 \pmod{n}$$

and

$$CD^2[f(pM-1)+1][f(m)^2-f(m)]f(n)\equiv 0 \pmod{n}.$$

These imply

$$C^2D^2[f(m)^2-f(m)]f(n)^2\equiv 0 \pmod{n}.$$

Assume that

$$f(m)^2 = f(m) \quad \text{for all} \quad m \in \mathbb{N}.$$

Since $f(n) \neq 0$ for all $n \in \mathbb{N}$, $(n, A) = 1$, the last relation implies

$$f(n) = 1 \quad \text{for all} \quad n \in \mathbb{N},\ (n, A) = 1.$$

Thus, $D = f(B) = 1$. Applying (3) with $(n, A) = 1$, $n \to \infty$, we get $1 = C + D = C + 1$, i.e. $C = 0$. This is a contradiction.

Thus, we have proved that there is a positive integer m_0 such that $M_0 := f(m_0)^2 - f(m_0) \neq 0$ and so

$$C^2 D^2 M_0 f(n)^2 \equiv 0 \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

This with the complete multiplicativity of f shows that

$$n | f(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

Lemma 2 is proved. ◊

Now we prove our theorem. By using Th. B and Lemma 1-2, the theorem is proved if $f(p) = 0$ for some prime number p.

In the next part we assume that $f(n) \neq 0$ for all $n \in \mathbb{N}$. We shall prove that either $f(n) :\equiv 1$ identicaly, or

$$f(n) \equiv 0 \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Assume the contrary, that $f(n) \not\equiv 1$ and that there exists a prime π such that

(6) $$(\pi, f(\pi)) = 1.$$

Let k be a positive integer. Then, we get from (3) the relations

$$f\left[ABk(k+1)n + B\right] \equiv C f(B) f(k) f(k+1) f(n) + D \pmod{n}$$

and

$$f\left[AB(k+1)n + B\right] \equiv C f(B) f(k+1) f(n) + D \pmod{n},$$

consequently

$$f\left[ABk(k+1)n + B\right] f\left[AB(k+1)n + B\right] \equiv$$
$$\equiv C^2 f(B)^2 f(k) f(k+1)^2 f(n)^2 +$$
$$+ CD f(B) f(k) f(k+1) f(n) + CD f(B) f(k+1) f(n) + D^2 \pmod{n}.$$

On the other hand, from (3), we have

$$f\left[ABk(k+1)n + B\right] f\left[AB(k+1)n + B\right] =$$
$$= f(B) f\left[A(k+1)^2 n(ABkn + B) + B\right] \equiv$$
$$\equiv C^2 f(B)^2 f(k) f(k+1)^2 f(n)^2 +$$
$$+ CD f(B) f(k+1)^2 f(n) + D f(B) \pmod{n}.$$

The last two relations imply

$$CDf(B)f(k+1)\,[f(k+1)-f(k)-1]\,f(n)\equiv \\ \equiv D(D-f(B)) \pmod n \tag{7}$$

for all $n\in\mathbb{N}$. Thus, for each $m\in\mathbb{N}$, we also have

$$CDf(B)f(k+1)\,[f(k+1)-f(k)-1]\,f(n)f(m)\equiv \\ \equiv D(D-f(B)) \pmod n,$$

and so

$$D(D-f(B))f(m)=D(D-f(B)) \quad \text{for all} \quad m\in\mathbb{N}.$$

Since $f(n)\not\equiv 1$, we have $D=f(B)$. Applying (7) with $n=\pi^s$, $s\in\mathbb{N}$, where π is the prime in (6), we have

$$CDf(B)f(k+1)\,[f(k+1)-f(k)-1]\equiv 0 \pmod{\pi^s}$$

for all $k,s\in\mathbb{N}$. Setting $s\to\infty$, the above relation gives

$$CDf(B)f(k+1)\,[f(k+1)-f(k)-1]=0$$

for all $k\in\mathbb{N}$. By our assumption, we have $Cf(B)f(k+1)\neq 0$, consequently $f(k+1)=f(k)+1$. Therefore $f(n)=n$ is satisfied for all $n\in\mathbb{N}$, which contradicts to (6).

Thus, we have proved that either $f(n)=1$ for all $n\in\mathbb{N}$ or $f(n)\equiv 0 \pmod n$ for all $n\in\mathbb{N}$.

In the first case we have $1=C+D$. In the second case, (3) and Th. B imply that there are a non-negative integer α and a real-valued Dirichlet character $\chi \pmod A$ for which $f(n)=\chi(n)n^\alpha$ for all $n\in\mathbb{N}$, $(n,A)=1$. It is clear that in this case $\alpha\neq 0$.

The proof of Th. is complete. $\Diamond$

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