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A FOUR-VERTEX THEOREM FOR SPACE CURVES

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Abstract: We prove a vertex theorem for space curves which need not lie on the boundaries of their convex hulls.

1. Introduction

A regular closed simple curve in Euclidean 3-space, lying on the boundary of its convex hull and without zero curvature points, has at least four points where the torsion τ vanishes. Under minor additional assumptions this was shown by Bisztriczky [2] and by Nuño Ballestros and Romero Fuster [13], and in full generality by Sedykh [21]. Admitting singular points and sign changes of the curvature κ, Romero Fuster and Sedykh [16] showed that

$$V + 2K + 3S \geq 4 \tag{1}$$

where S is the number of singular points, K the number of zeros of κ, and V the number of zeros of τ. Here we will consider regular closed space curves which may have double points and need not lie on the boundaries of their convex hulls. We show that

(2) $$V + K + D \geq 4$$

where D is the number of extrema of the conical curvature τ/κ. We call such points *Darboux vertices* because there the Darboux vector changes its sense of rotation within the rectifying plane. For V and K we only count sign changes of τ and κ.

2. The vertex theorem

Definition 1. By a *closed space curve* we mean an immersion $X : S^1 \to$ $\to E^3$ of class C^3 with the following properties

(P_1) d^2X/ds^2 (s arc length) vanishes only at finitely many points and the unit normal vector $N = \kappa^{-1}d^2X/ds^2$ may locally be defined as a C^1 vector field,

(P_2) curvature κ and torsion τ never vanish at the same point,

(P_3) $X(S^1)$ does not lie in a plane (τ in not identically zero).

Remark 1. Usually $\kappa \neq 0$ is assumed for convenience. But it suffices that locally there is a C^1 Frenet frame. This has been worked out by Nomizu [12], see also Grüß [9], Fenchel [7], and Randrup and Røgen [15]. Nomizu shows that (P_1) is fulfilled by C^∞ curves if for every point there is some $k \in \mathbb{N}$ such that $d^kX/ds^k \neq 0$. By (P_1) we allow the curvature κ to change its sign. But we have to pay attention to the fact that the frame cannot be defined globally in a unique way if κ has an odd number of sign changes: after having run through the curve once, the normal vector will have changed its direction. Another strange fact may show up in examples: if curves with $\kappa \neq 0$ approach a curve with a point where κ changes sign, then the torsion may change discontiuously. This was noted by Călugăreanu [4, 616]. See also Section 4.4 and Fig. 4.

Remark 2. Assumption (P_2) simplifies the situation, but it excludes some interesting examples. Consider for instance a Möbius strip. It is the rectifying strip of its middle line. This curve must have at least one point where κ changes sign. But there τ will vanish too. For if at some point $\kappa = 0$, $\tau \neq 0$, then the cuspidal edge of the rectifying strip touches the curve. This follows from the following representation of the cuspidal edge

(3) $$X + \frac{\kappa}{\kappa'\tau - \tau'\kappa}(\tau T + \kappa B),$$

where T is the unit tangent vector, B the binormal vector (Scheffers [8,417]). It also follows from Th. 3 of Randrup and Røgen [15] that for an embedded rectifying strip $\tau = 0$ at points where $\kappa = 0$. See also Section 4.1.

Probably the Th. remains true, if (P_2) is replaced by the weaker condition that $\lim \tau/\kappa$ exists in $\mathbb{R} \cup \{\infty, -\infty\}$, possibly without counting points twice.

Remark 3. Notice that we do not exclude double points.

Definition 2. A point where τ or κ changes sign is called a *vertex* or an *inflection*, respectively. A point where τ/κ or κ/τ has a local strong extremum is called a *Darboux vertex.* For a given closed curve we denote the numbers of vertices, inflections, and Darboux vertices by V, K, and D respectively. We admit the possibility that a sign change takes place in such a way that τ or κ vanishes in an interval which is then to be counted as one point, and we make a similar convention for D.

Together wit the curve X we will consider its tangent image $T = X'$ as a curve on the unit sphere S^2, the prime denoting differentiation with respect to arc length. Its geodesic curvature is τ/κ (Frenchel [7]). We call the tangent circles on S^2 with radius κ/τ *osculating circles of* T.

Lemma 1. *If X is a closed space curve, then $T(S^1)$does not lie in any closed hemisphere.*

Proof. Assume for instance that $T(S^1)$ lies in the closed upper hemisphere. Then its third component $T^3 \geq 0$ and not always 0 since X is not a plane curve. Integrating T with respect to arc length of X shows that X could not be closed. (The converse of Lemma 1 is less trivial. It has been proved by Fenchel [6]).◊

Lemma 2. *The osculating circles of an arc on S^2 of class C^2 with strictly monotone geodesic curvature have the nesting property, i.e. they are contained in one another without meeting. Conversely, from the nesting property follows the strict monotonity.*

Proof. This property is well-known for plane curves. By stereographic projection we can defer it to S^2 in the following way (see also Weiner [24, 431]). Jackson [11] characterizes arcs of class C^2 on surfaces with strictly monotone geodesic curvature by the fact that they cross their osculating curvature circles at every point. If we project an arc on S^2 with strictly monotone geodesic curvature stereographically into the plane, this property remains true. The image thus has strictly monotone curvature and their osculating circles have the nesting property. The osculating circles of the pre-image then have the same property. The other direction follows similarly. Note that the proof of the nesting property in the plane does not need the differentiability of the curvature radius as was shown by Ostrowski [14, 325]. — It is also possible to remodel the proof for planar arcs on the sphere. ◊

Theorem. *For a closed space curve* (in the sense of Def. 1) *we have*

$$\sum := V + K + D \geq 4. \tag{4}$$

Corollary. *For a closed space curve with $\kappa \geq 0$, $\tau \geq 0$ we have $D \geq 4$.*
Remark 4. Under additional hypotheses, Takasu [23] has shown the statement of the Cor. because his "dual curvature" is κ/τ. Jackson's critique [10, 810], does only refer to Takasu's similar theorem for spherical curves. See also Section 4.3.
Proof of the Theorem. We treat the cases

$$K = 0,\ K = 1,\ K \geq 2$$

separately. V is an even number, since τ is continuous and periodic.

(a_1) $K = 0$ and $V = 0$. This is the case of the Cor.. We have to show $D \geq 4$. The tangent image T does not have constant geodesic curvature τ/κ because otherwise T would be a circle, but not a great circle since we exluded $\tau \equiv 0$. Then the curve X would not be closed by Lemma 1. Assume now that τ/κ has only two extrema which then divide T into two arcs of monotone non-constant geodesic curvature. Let C_0 be the osculating circle of T at a point with minimal geodesic curvature. It follows from Lemma 2 that, for both arcs of T, all osculating circles lie on one side of C_0, and if C_0 is not a great circle, they lie on the smaller side since τ/κ is monotone. Then $T(S^1)$ lies on this same side of C_0, and by Lemma 1 the curve X cannot be closed. This exludes $D = 2$. Since $K = 0$, the curvature κ of X is uniquely defined and therefore periodic. Therefore τ/κ is periodec too, D is even, and $\sum = D \geq 4$.

(a_2) $K = 0$ and $V \geq 2$. By (P_2) τ/κ vanishes at the zeros of τ. Because there are at least two of them, there will be at least two extrema of τ/κ between them, possibly of infinite value. Note that κ may be zero but does not change sign. This shows $D \geq 2$ and $\sum = V + D \geq 4$.

(b) $K = 1$. Consider the arc A wich remains if the point or interval where κ changes sign is taken away from the curve X. The sign of κ does not change on A and τ changes sign, if at all, an even number of times. Therefore τ/κ has the same sign near both end points of A, $\tau/\kappa \to \infty$, say. Then there must be a minimum of τ/κ on A, possibly wit value $-\infty$. Thus $D \geq 1$. Furthermore D is odd. We will now derive a contradiction from $D = 1$ and $V = 0$. This will then show $D \geq 3$ or $V \geq 2$. Since $K = 1$ and $D \geq 1$, we then have $\sum = V + K + D \geq 4$. Indeed, if $D = 1$ and $V = 0$, let p be the point, or one of the points, where τ/κ has the extremum. A is divided by p into two subarcs A_1 and A_2 with monotone τ/κ which, by Lemma 2, lie on the same side of the osculating circle $C(p)$

at p. $C(p)$ may be a great circle (if $\tau = 0$ at p) but all the other ones are smaller circles or coincide with $C(p)$ because they shrink to points at the other ends of A_1 and A_2. Thus A_1 and A_2, and therefore X, lie in a hemisphere which is impossible by Lemma 1.

(c) $K \geq 2$. We consider a subarc between successive sign changes of κ. It must contain an extremum of τ/κ or sign change of τ/κ, that is a sign change of τ. Since there are at least two such subarcs, we have $V + D \geq 2$ and $\sum = V + K + D \geq 4$. $\Diamond$

3. Examples for the equality cases

The proof exhibits 6 different cases with $\sum \geq 4$. We show by examples that in all these cases $\sum = 4$ is possible. Four of them are $(1,k)$-torus curves

$$(5) \qquad \big((R + \cos kt)\cos t, (R + \cos kt)\sin t, \sin kt\big).$$

Corresponding part of proof	K	V	D	Examples of curve
(a_1)	0	0	4	$(1,2)$-torus curve with $R = 4$
(a_2)	0	2	2	$(1, 1$-torus curve with $R = 1$
(b_1)	1	0	3	$(1,2)$-torus curve with $R = 5$, modified
(b_2)	1	2	1	$(1, 1$-torus curve with $R = 2$
(c_1)	2	0	2	$(1,2)$-torus curve with $R = 5$
(c_2)	2	2	0	curve with $\kappa = \sqrt{3}\cos t, \tau = \sqrt{3}\sin t$

According to Prop. 1 of Costa [5] we can choose any R with $3 <$ $< R < 5$ in (a_1).

In (a_2) we could also choose $R = 0$ which gives Viviani's curve, the intersection of the unit sphere with the cylinder $(x - 1/2)^2 + y^2 = 1/4$. This curve is well-known in architecture. It has a double point which is allowed in our Th..

In (b_1) the curve in (c_1) is modified in such a way that one of the inflections is removed and that the other one stays. It has the representation

$$(6) \quad \big((5 - \tfrac{1}{3}(1 - \sin t)^2 + \cos 2t)\cos t, (5 - \tfrac{1}{3}(1 - \sin t)^2 + \cos 2t)\sin t, \sin 2t\big).$$

In (c_2) we have one of Scofield's [20] curves of constant precession, namely

$$(7)\qquad \Big(\frac{3}{4}\sin t-\frac{1}{12}\sin 3t, -\frac{3}{4}\cos t-\frac{1}{12}\cos 3t, \frac{\sqrt{3}}{2}\sin t\Big).$$

In this last example t is the arc length.

Obviously example (c_2) has the properties claimed in the table. In all other cases it is not much work to generate plots of τ, κ, τ/κ and to convince oneself that the claims are true. It is also possible to transform the numerators of τ, κ, τ/κ into polynomials of $\sin t$ or $\cos t$ and to determine the exact number of zeros by means of Sturm sequences. This has been worked out using Maple V.

In Sedykh's theorem and its extension mentioned in the introduction it is assumed that the curve lies on the boundary of its convex hull. This property could not strengthen our Th. as examples (b_2), (c_1) and (c_2) show. These curves lie on the boundaries of their convex hulls, which easily follows from the convexity of their projections onto the xy-plane.

4. Further remarks

4.1. Geometric interpretation of Darboux vertices

Darboux vertices were defined in Def. 2 as points were the conical curvature τ/κ has an extremum, including the case that this extremum is $+\infty$ or $-\infty$. The conical curvature really has no simple interpretation. In the proof we used the fact that these points correspond to vertices of the tangent image (extrema of its geodesic curvature). At the end of the introduction we mentioned a geometric interpretation of Darboux vertices which we will now explain in greater detail.

At each point of the curve X the plane spanned by the tangent vector T and the binormal vector B touches the rectifying strip (compare Remark 2). It is called so because the curve becomes a straight line if the strip is rolled out on a plane. Like any developable surface, it is generated by a family of straight lines, which here have the directions of the Darboux vectors $\tau T+\kappa B$. So, if one wraps a rectangular strip along the curve X, then one can approximately see where the generating straight lines change their sense of rotation with respect to the bases $\{T, B\}$, thus identifying the Darboux vertices.

4.2. Explanation of the figures

Fig. 1 shows the curve (b_2) on the torus. The inflection, where κ changes its sign, is marked by a small cube.

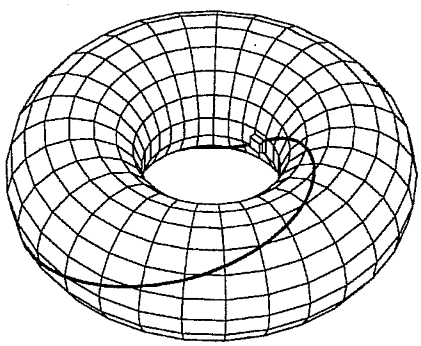

Figure 1

Fig. 2 shows this curve together with its rectifying strip and the generators of the strip. An edge of regression touches the curve at the inflection (compare Remark 2). Next to the inflection, 2 vertices are marked. Here the generators are orthogonal to the curve. Opposite the inflection a Darboux vertex is marked. All this can more easily be seen when the rectifying strip is rolled out on the plane (Fig. 3): a Darboux vertex at 0, vertices at $\pm\pi/3$, and an inflection at π. Strictly speaking Fig. 3 does not show the rectifying strip rolled out since we did not introduce the arc length parameter.

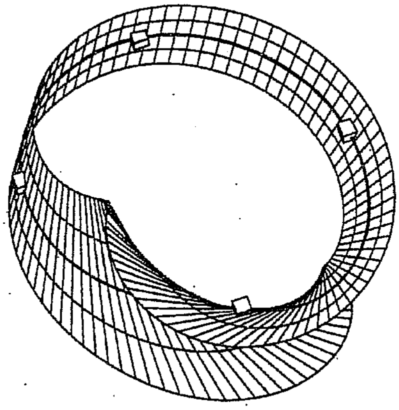

Figure 2

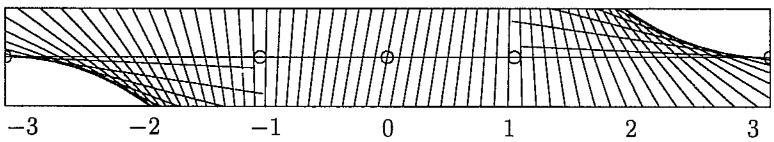

Figure 3

Fig. 4 shows the torsion of the curve together with the torsions of 4 approaching curves ($R = 1.6, 1.995, 2, 2.005, 2.2$). It illustrates the discontinuous behavior of the (continuous) torsion when an inflection is approached by curves without inflections (compare Remark 1 and Section 4.4). This may be made plausible in the following way. During the approximation the normal vectors $\kappa^{-1}d^2X/ds^2$ have to approach a normal vector which is discontinuous at the inflection if it is defined in the traditional way. They have thus to perform a rotation of about 180°. This gives high values to the torsion and, depending on the sense of rotation, 2 additional zeros of τ or none.

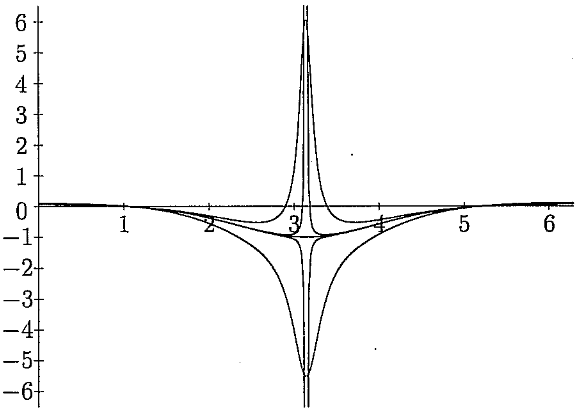

Figure 4

4.3. The 4-vertex theorem for spherical curves

In order to proof the Cor. (part a_1 of the proof) one could be temped to use the four-vertex theorem for spherical curves. But this is impossible: for $\kappa > 0$ the tangent image T of X is a curve immersed into the sphere S^2 which does not lie in a hemisphere (Lemma 1). If it would be free of double points, then it would have points of inflection (Arnold [1, 54]), Blaschke [3,49], ex. 25), and this would contradict $\tau/\kappa \geq 0$. But curves on S^2 with double points need not have 4 extrema of the geodesic curvature τ/κ.

Saban [17, 257] considers closed space curves X with $\kappa > 0$, $\tau > 0$ and with the property that the binormal B describes an oval on S^2. He proves a four-vertex-theorem for ovals on the sphere and thus concludes the existence of 4 Darboux vertices of X. He calls them *helicoidal points.* This seems to be a special case of our Cor., but is in fact a void assertion because B cannot be an oval. This follows from the fact that T cannot be an oval (as explained above) because T and B are polars of each other (Fenchel [7, 46]). For a closed space curve with $\kappa > 0$, $\tau > 0$ Saban gives the following example of Segre [22]

$$(8)\quad X = \big(t/\omega, (t^2-1)/\omega, t^3/\omega\big),\ \omega = t^4 - 3t^2/2 + 1,\quad -\infty \le t \le \infty.$$

Its tangent image T is not an oval since it has 2 double points, namely $\dot{X}(t_i) = \dot{X}(-t_i)$, $i = 1, 2$, for $t_1 = \sqrt{1 + 1/\sqrt{2}}$ and $t_2 = \sqrt{1 - 1/\sqrt{2}}$. Thus T and B are not ovals.

4.4. Gericke's example

Gericke [8] used $(2, 1)$-torus curves

$$(9)\quad \big((R - \sin\frac{t}{2})\cos t, (R - \sin\frac{t}{2})\sin t, \cos\frac{t}{2}\big),\ 0 \le t \le 4\pi,\ R > 1,$$

in order to show that closed space curves need not have more than 2 curvature extrema. For $R = 5/4$ we obtain another example for (b_2).

Gericke concludes from a result of Scherk [19, 763] on curves of order 4 that such torus curves have exactly 2 zeros of the torsion τ for all values of R. But for $5/4 < R < 2$ they have 4 of them. The reason is that these curves are not of order 4 for all R. This can be seen as follows: for $R = 5/4$ the curvature κ changes sign at $t = \pi$. The osculating plane locally supports the curve at this point. Thus, for 3 values of t approaching π there is a 4th one also approaching π such that the corresponding points lie on a plane, and there are 2 other points of the curve not close to the inflection point. The curve with $R = 5/4$ is therefore of order (at least) 6. Nevertheless it has only 2 sign changes of τ. But for $5/4 < R < 2$ there are 4 sign changes of τ, and we can conclude that for these values of R the order must be at least 6. The existence of 2 additional sign changes of τ may also be explained as in Section 4.2. By the way, Scherk [19] showed that $V = 4$ for a curve of order 4 lying on the boundary of its convex hull. As mentioned in the introduction, Sedykh proved this without assuming 4th order. In several recent papers, with reference to [19], this is called *Scherk's Conjecture*. But G. Thorbergsson pointed out to me that such a conjecture is not expressed in [19].

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