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MEASURABLE SOLUTIONS OF FUNCTIONAL EQUATIONS SATISFIED ALMOST EVERYWHERE

Antal **Járai**

Department of Numerical Analysis, Eötvös L. University, H–1518 Budapest, Pf. 32, Hungary

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Abstract: In this paper under certain conditions it is proved that if the functional equation

$$H(x, y, f_1(G_1(x,y)) \dots , f_n(G_n(x,y))) = 0$$

with measurable unknown functions $f_1, \dots , f_n$ is satisfied for almost all pair (x, y) from an open set E then there exist (unique) continuous functions $\tilde{f}_i$ such that $\tilde{f}_i = ccf_i$ almost everywhere for $i = 1, \dots , n$ and replacing f_i by $\tilde{f}_i$ for $i = 1, \dots , n$ the above equation is satisfied everywhere on E.

In 1960 P. Erdős raised the following problem: Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the relation

$$f(x + y) = f(x) + f(y)$$

for almost all $(x, y) \in \mathbb{R}^2$ in the sence of the planar Lebesgue measure. Does there exists an additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ almost everywhere in the sence of the linear Lebesgue measure? A positive answer was given to this question by N. G. de Bruijn, and independently, by W. B. Jurkat. For further references, see the book [9] of Kuczma, pp. 443–447.

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Here we deal with a related question. Suppose that the more general functional equation

$$(1)\qquad H(x,y,f(G(x,y)),f_1(G_1(x,y)),\dots,f_n(G_n(x,y)))=0$$

with unknown measurable functions $f,f_1,f_2,\dots,f_n$ is satisfied for all (x,y) from a subset E' of some open subset E of $\mathbb{R}^r\times\mathbb{R}^k$, for which $E\setminus E'$ has Lebesgue measure zero in $\mathbb{R}^r\times\mathbb{R}^k$. Here all functions are supposed to be defined on some open subset of some Euclidean space, and taking values in some Euclidean space. The known functions H and $G,G_1,\dots,G_n$ supposed to be smooth. Is it possible to find functions such that $\tilde{f}=f$ and $\tilde{f}_i=f_i$ almost everywhere for $i=1,\dots,n$ such that replacing $f,f_1,\dots,f_n$ with $\tilde{f},\tilde{f}_1,\dots,\tilde{f}_n$, respectively, the functional equation (1) is satisfied everywhere on E?

We shall prove that under reasonable conditions the answer is yes and the functions $\tilde{f}$, $\tilde{f}_i$, $i=1,\dots,n$ are continuous. Of course, this is related to earlier results in Járai [4], [5], [6] proving that measurable solutions of a functional equation satisfied everywhere on an open set are continuous.

Under certain condition equation (1) can be reduced to a simpler explicit equation. Namely, suppose, that the term $f(G(x,y))$ can be expressed from equation (1), and it is posssible to introduce a new variable $t=G(x,y)$ instead of x to obtain the functional equation

$$(2)\qquad f(t)=h(t,y,f_1(g_1(t,y)),\dots,f_n(g_n(t,y))),\quad (t,y)\in D'.$$

Here D' is the image of E' by the mapping $(x,y)\mapsto(G(x,y),y)$. We suppose that this mapping is a diffeomorphism mapping the open set E onto the open set D, hence this mapping and its inverse carry sets having Lebesgue measure zero into sets having Lebesgue measure zero. Now, if we are able to prove that there is subset T' of the domain T of f such that $T\setminus T'$ has measure zero and $f|T'$ can be (uniquelly) extended to a continuous function $\tilde{f}$ defined on T, then the functional equation (2) is satisfied with $\tilde{f}$ instead of f almost everywhere on D, namely, on the set $D_0'=D'\cap(T'\times\mathbb{R}^k)$. This means that if we replace f by $\tilde{f}$ then equation (1) is still satisfied, at least on some subset E_0' of E_0 for which $E\setminus E_0'$ has measure zero. Now, repeating this process with f_1, we may obtain a subset E_1' of E_0' and a function $\tilde{f}_1$ continuous and equal almost everywhere to f_1 such that (1) is satisfied if f and f_1 are replaced by $\tilde{f}$ and $\tilde{f}_1$, respectively. Finally, we obtain a subset $E''=E_n'$ such that $E\setminus E''$ has measure zero and (1) is still satisfied if we replace $f,f_1,\dots,f_n$ by the continuous function $\tilde{f},\tilde{f}_1,\dots,\tilde{f}_n$ equal almost everywhere to them.

Since the left hand side of (1) is continuous on E and equal to zero on a dense subset $E'' \subset E$, we obtain that equation (1) is satisfied on E if $f, f_1, \ldots, f_n$ are replaced by $\tilde{f}, \tilde{f}_1, \ldots, \tilde{f}_n$, respectively.

Summarising, our problem concerning equation (1) can be reduced to the following problem: Suppose that equation (2) is satisfied on a subset D' of the open set D for which $D \setminus D'$ has measure zero. Give reasonable conditions under which there is an appropriate subset T' of the domain T of f such that $T \setminus T'$ has measure zero and the function $f|T'$ has a (unique) continuous extension $\tilde{f}$ to T. We shall see that the set T' can be choosen to be the set of all points $t \in T$ for which the set $\{y : (t, y) \in D \setminus D'\}$ has measure zero. By Fubini's theorem, $T \setminus T'$ has measure zero. More generally, $\tilde{f}$ is determined by any subset of this T' which is still dense in T. This will be proved in a much more general setting in Th. 2. The proof use Th. 1, which is a refinement of a part of Th. 2.6 in Járai [7]. For the sake completeness, we shall give the proof of Th. 1, too. The case of Euclidean spaces will be obtained in Th. 3 as a consequence of Th. 2.

We shall denote by $\mathbb{R}$ the set of real numbers. The norm on $\mathbb{R}^n$ is denoted by $|\ |$. If f is a function, $\operatorname{dom} f$ denote the domain of f. If $D \subset X_1 \times X_2 \times \ldots \times X_n$, we shall use the *partial sets*

$$D_{x_i} = \big\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) : (x_1, \ldots, x_n) \in D\big\}.$$

The *partial functions* $f_{x_i} : D_{x_i} \to Y$ are defined by

$$f_{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_n)$$

whenever $(x_1, \ldots, x_n) \in D$. Also $D_{x_{i_1}, \ldots, x_{i_r}}$ and $f_{x_{i_1}, \ldots, x_{i_r}}$ are defined similarly. Now, if X_i and Y are normed spaces and $D_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$ is an open subset of X_i we define the *partial derivative* denoted by $\dfrac{\partial f}{\partial x_i}$ as the derivative of $f_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$ (if it exists).

We shall use the terminology and notations of Bourbaki [2] concerning topology. Hence, every regular space is supposed to be Hausdorff. Note that the limit of a function at a point is defined by Bourbaki without excluding this point.

Concerning measure theory, we follow the terminology of Federer [3]. Hence a *measure* means a countably subadditive extended real valued nonnegative function μ defined on all subsets of a set; this is called *outer measure* in other terminology. The set of measurable sets is defined by the Charatheodory condition: a set $A \subset X$ is called μ *measurable* if $\mu(T \cap A) + \mu(T \setminus A) = \mu(T)$ for every $T \subset X$. The measure μ is called *finite* if $\mu(X) < \infty$; it is called σ-*finite*, if X can be represented as the countable union of measurable sets having finite μ measure. λ^n will denote the Lebesgue measure on $\mathbb{R}^n$. Let μ be a measure on X and Y a topological space. A *function* is called μ *measurable* on a set A if its domain contains almost all of A, its range is in the topological space Y, and if $A \cap f^{-1}(V)$ is μ measurable whenever V is an open subset of Y. In particular A have to be measurable.

Our terminology concerning topological measures is a somewhat different from the terminology of Federer's book. By a *Radon measure* we mean a measure μ defined on a Hausdorff space X, with the following properties:

(1) the μ measure of any compact subset K of X is finite;

(2) every open subset V is measurable and

$$\mu(V) = \sup\{\mu(K) : K \subset V,\ K \text{ compact}\};$$

(3) if A is any subset of X, then

$$\mu(A) = \inf\{\mu(V) : A \subset V,\ V \text{ open}\}.$$

It is not hard to prove that if μ is a Radon measure then

$$\mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ compact}\}$$

whenever A is a μ measurable set with finite μ measure.

Let μ be a Radon measure on the Hausdorff space X and let Y be a topological space. Let f be a function mapping almost all of a subset E of X into Y. The function f is called a *Lusin μ measurable function on E*, if for each measurable subset A of E having finite measure and for each $\varepsilon > 0$ there is a compact subset C of A such that $\mu(A \setminus C) < \varepsilon$ and $f|C$ is continuous. In this setting Lusin's theorem reads as follows:

Lusin's Theorem. *Let μ be a Radon measure on the Hausdorff space X, let E be a μ measurable subset of X and let Y be a topological space. A function f mapping a subset of X into Y which is Lusin μ measurable on E, is μ measurable on E. Conversely, if f is μ measurable on E and the topology of Y has a countable base, then f is Lusin μ measurable on E.*

Proof. See Oxtoby [9], 8.2.◊

Theorem 1. *Let T be a topological space, X and Y Hausdorff spaces with Radon measures μ and ν, respectively. Suppose that ν is σ-finite and $g : T \times Y \to X$ is a continuous function with the following property:*

(1) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $t \in T$, $B \subset Y$, $\nu(B) \geq \varepsilon$ then $\mu(g_t(B)) \geq \delta$.*

Suppose, moreover, that $t_0 \in T$ and f is a Lusin ν measurable function with values in a topological space Y on some measurable set D containing $g_{t_0}(Y)$. Then for every sequence t_m from T convergent to t_0 has a *subsequence* t_{m_k} *such that* $f(g(t_{m_k}, y)) \to f(g(t_0, y))$ *for almost all* $y \in Y$ *as* $k \to \infty$.

Proof. Let us observe that whenever K' is a compact subset of X and $C' = g_{t_0}{}^{-1}(K')$ has finite ν measure, then for each $\varepsilon > 0$ there exists a neighbourhood T_0 of t_0 such that for each $t \in T_0$ we have $\nu(C' \setminus g_t^{-1}(K')) < < \varepsilon$. To prove this, let us choose a compact subset C'' of the Borel set C' for which $\nu(C' \setminus C'') < \varepsilon/2$ and let $K'' = g_{t_0}(C'')$. Let us choose an open set V containing K'' such that $\mu(V \setminus K'') < \delta$, where δ corresponds to $\varepsilon/2$ by (1). For each $y \in C''$ there exist open neighbourhoods Y_y and T_y of y and t_0, respectively, such that $g(T_y \times Y_y) \subset V$. Let us choose a finite subcovering $Y_{y_1}, \dots, Y_{y_n}$ of the covering Y_y, $y \in C''$, and let $T_0 = \cap_{i=1}^{n} T_{y_i}$. Then for $t \in T_0$ the set $C'' \setminus g_t^{-1}(K'')$ is mapped by g_t into $V \setminus K''$, hence has ν measure less then $\varepsilon/2$. Now, since $K'' \subset K'$ and

$$C' \setminus g_t^{-1}(K') \subset (C' \setminus C'') \cup (C'' \setminus g_t^{-1}(K''))$$

we obtain that $\nu(C' \setminus g_t^{-1}(K')) < \varepsilon$.

First we shall prove the statement for a compact subset C of Y instead of Y. So let C be a compact subset of Y and $K = g_{t_0}(C)$, moreover let $t_m \to t_0$ be a sequence in T. Let $\varepsilon_i = 2^{-i}$ and let $\delta_i > 0$ be the corresponding sequence of numbers δ by (1). Let us choose a compact subset $K_1 \subset K$ such that $f|K_1$ is continuous and $\mu(K \setminus K_1) < \delta_1$ and let $C_1 = g_{t_0}^{-1}(K_1)$. Then $\nu(C \setminus C_1) < \varepsilon_1$. By induction, using what we proved in the previous paragraph, we may find a sequence of indices $m_1 < m_2 < \dots$ such that $\nu(C_1 \setminus g_{t_j}^{-1}(K_1)) < \varepsilon_{i+1}$ whenever $j \geq m_i$. This implies that $\nu(C_1 \setminus \cap_{r=1}^{\infty} g_{t_{m_r}}^{-1}(K_1)) < \varepsilon_1$. Now let K_2 be a compact subset of K such that $f|K_2$ is also continuous and $\mu(K \setminus K_2) < \delta_2$. Let $C_2 = g_{t_0}^{-1}(K_2)$, then $\nu(C \setminus C_2) < \varepsilon_2$. Let us apply induction again, but using the new subsequence instead of the original sequence. Then we obtain a subsequence such that $\nu(C_2 \setminus \cap_{s=1}^{\infty} g_{t_{m_{r_s}}}^{-1}(K_2)) < \varepsilon_2$. Continuing this process and taking the diagonal sequence, we arrive at a subsequence t_{m_p} of t_m such that the measure of the set

$$E_i = (C \setminus C_i) \cup \left((\cup_{t=i}^{\infty}(C_i \setminus g_{t_{m_p}}^{-1}(K_i)))\right)$$

is less then $2\varepsilon_i$. Now let $E = \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i$. Clearly, $\nu(E) = 0$. If $y \in C \setminus E$ then there exists a k such that $y \notin E_i$ for $i \geq k$. This means on one hand that $y \notin C \setminus C_i$ for $i \geq k$, that is, $y \in C_i$ for $i \geq k$. This implies that $g_{t_0}(y) \in K_i$ for $i \geq k$, in particular, $g_{t_0}(y) \in K_k$. On the other hand, if $i \geq k$ then for each $p \geq i$ we have $y \notin C_i \setminus g_{t_{m_p}}^{-1}(K_i)$. We shall apply this only for $i = k$ to obtain that $g_{t_{m_p}}(y) \in K_k$ whenever $p \geq k$. Because $f|K_k$ is continuous, we obtain that $f(g_{t_{m_p}}(y)) \to f(g_{t_0}(y))$. The general case can be obtained representing Y as the union of a σ-compact set and a set of measure zero and using the diagonal process. $\Diamond$

Theorem 2. *Let T and Z_i $(i = 1, 2, \dots, n)$ be topological spaces, Z, Y and X_i $(i = 1, 2, \dots, n)$ Hausdorff spaces. Suppose that $D \subset T \times Y$, $T' \subset T$, $X_i' \subset X_i$. Let $f : T' \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \dots, n)$, $f_i : X_i' \to Z_i$ $(i = 1, 2, \dots, n)$ and $h : D \times Z_1 \times \dots \times Z_n \to Z$ be functions, ν a Radon measure on Y, μ_i a Radon measure on X_i and $\mu_i(X_i \setminus X_i') = 0$ $(i = 1, 2, \dots, n)$. Suppose that $t_0 \in T$ has countable base of neighbourhoods and the following conditions are satisfied:*

(1) *for each fixed y in Y, the function h is continuous in the other variables;*

(2) *f_i is Lusin μ_i measurable on the measurable subset A_i of X_i $(i = 1, 2, \dots, n)$;*

(3) *g_i is continuous on D $(i = 1, 2, \dots, n)$;*

(4) *there exist sets V and K such that V is open, K is compact, $V \times K \subset D$, $t_0 \in V$, $\nu(K) > 0$ and $K \subset \cap_{i=1}^n g_{i,t_0}^{-1}(A_i)$;*

(5) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $B \subset K$ and $\nu(B) \geq \varepsilon$ implies $\mu_i(g_{i,t}(B)) \geq \delta$ whenever $1 \leq i \leq n$ and $t \in V$;*

(6) *there exist a subset V' of $T' \cap V$ such that t_0 is contained in the closure of V' and for each $t \in V'$ for almost all $y \in K$ we have*

$$f(t) = h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))).$$

Then $\lim_{t \in V', t \to t_0} f(t)$ exists.

Proof. Let t_m be a sequence in V' convergent to t_0. Applying Th. 1 for g defined by $g(t, y) = g_1(t, y)$ whenever $(t, y) \in V \times K$ and for the restriction of the measure ν to the subsets of K we obtain a subsequence t_{m_k} for which $f_1(g_1(t_{m_k}, y)) \to f_1(g_1(t_0, y))$ for almost all $y \in K$. Now repeting this with f_2, g_2 and the subsequence t_{m_k}, we obtain a sub-subsequence, etc. Finally, we obtain a subsequence t_{m_l} of the original sequence t_m such that for almost all $y \in K$ we have $f_i(g_i(t_{m_l}, y)) \to$ $\to f_i(g_i(t_0, y))$ for $1 \leq i \leq n$. Fixing any such $y \in K$, by the functional equation and the properties of h this means that each sequence $t_m \to t_0$ in T has a subsequence t_{m_l} for which $f(t_{m_l})$ converges. Of course, its limit $z_0 \in Z$ does not depend on y. Hence, for all y from a set $K' \subset K$ for which $\nu(K \setminus K') = 0$ we have

$$z_0 = h(t_0, y, f_1(g_1(t_0, y)), \dots, f_n(g_n(t_0, y))). \tag{7}$$

Suppose that it is not true that $\lim_{t \in V', t \to t_0} f(t) = z_0$. Then there exists a neighbourhood W of z_0 such that there is no neighbourhood U of t_0 for which the set $U \cap V'$ is mapped by f into W. If U_m, $m = 1, 2, \dots$ is a countable neighbourhood base of t_0, then let us choose a sequence

$t'_m \in U_m \cap V'$ for which $f(t_m) \notin W$. Repeating the above process with t'_m instead of t_m, we obtain a subsequence t'_{m_j} such that for almost all $y \in K$ we have $f_i(g_i(t'_{m_l}, y)) \to f_i(g_i(t_0, y))$. Now, if we choose an $y \in K'$ for which the above sequences converge for $i = 1, 2, \dots, n$, then using the functional equation and (7) we obtain a contradiction. ◊

To prove the next theorem we need a lemma.

Lemma 1. *Let Y be an open subset of $\mathbb{R}^k$, T a topological space, D an open subset of $T \times Y$ and $(t_0, y_0) \in D$. Suppose, that the function $g : D \to \mathbb{R}^r$ is continuous and continuously differentiable with respect to y. If the rank of the matrix $\frac{\partial g}{\partial y}(t_0, y_0)$ is r then there exist neighbourhoods T^* and Y^* of t_0 and y_0, respectively for which*

(1) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda^r(g_t(B)) \geq \delta$ whenever $t \in T^*$, $B \subset Y^*$ $\lambda^k(B) \geq \varepsilon$;

(2) if A is a λ^r measurable subset of $\mathbb{R}^r$ then $g_t^{-1}(A) \cap Y^*$ is a λ^k measurable subset of Y for each $t \in T^*$.

Proof. This is Lemma 3.2 in Járai [6]. ◊

Theorem 3. *Let Z be a regular space, Z_i $(i = 1, 2, \dots, n)$ topological spaces and T a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$ $(i = 1, 2, \dots, n)$ and D an open subset of $T \times Y$. Let $T' \subset T$ be a dense subset, $f : T' \to Z$, $g_i : D \to X_i$ and $h : D \times Z_0 \times Z_1 \times \dots \times Z_n \to Z$. Suppose that the function f_i is almost everywhere defined on X_i with values in Z_i $(i = 1, 2, \dots, n)$ and the following conditions are satisfied:*

(1) for all $t \in T'$ for almost all $y \in D_t$

$$f(t) = h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y)));$$

(2) for each fixed y in Y, the function h is continuous in the other variables;

(3) f_i is λ^{r_i} measurable $(i = 1, 2, \dots, n)$;

(4) g_i and the partial derivative $\frac{\partial g_i}{\partial y}$ is continuous on D $(i = 1, 2, \dots, n)$;

(5) for each $t \in T$ there exists a y such that $(t, y) \in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has rank r_i at $(t, y) \in D$ $(i = 1, 2, \dots, n)$.

Then f has a unique continuous extension to T.

Proof. We shall reduce this theorem to the previous theorem. We shall prove that the limit $\lim_{t \in T', t \to t_0} f(t)$ exist for each $t_0 \in T$. If this is proved, then defining the extension of f to T by this limit, the theorem is proved.

Let us choose for a given t_0 a y_0 such that $(t_0, y_0) \in D$ and the partial derivative $\dfrac{\partial g_i}{\partial y}$ has rank r_i at (t_0, y_0) $(i = 1, 2, \dots, n)$. To prove that the limit exists, we will replace D by a suitable smaller set D^*. By the previous lemma there exist open neighbourhoods V and W of t_0 and y_0, respectively, with the following properties:

(6) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lambda^{r_i}(g_{i,t}(B)) \geq \delta$ whenever $1 \leq i \leq n$, $t \in V$, $B \subset W$ and $\lambda^k(B) \geq \varepsilon$;

(7) $g_{i,t}^{-1}(\operatorname{dom} f_i) \cap W$ is a λ^k measurable subset of $\mathbb{R}^k$ whenever $1 \leq i \leq n$ and $t \in V$;

(8) $\overline{W}$ is a compact set, $V \times \overline{W} \subset D$ and

$$\lambda^k\Big(\bigcap_{i=1}^{n} g_{i,t_0}^{-1}(\operatorname{dom} f_i) \cap W\Big) > 0.$$

Let $D^* = V \times W$, $h^* = h|D^* \times Z_1 \times \dots \times Z_n$, $g_i^* = g_i|D^*$ $(i = 1, 2, \dots, n)$, and let us apply the previous theorem on the set D^* instead of D. $\Diamond$

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