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ON A FUNCTIONAL EQUATION OF HOSSZÚ TYPE

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Abstract: The Hosszú type functional equation
$$f(x+y-x\circ y)+f(x\circ y)=f(x)+f(y)$$
is solved, where $x\circ y:=\ln(e^x+e^y)$ $(x,y\in\mathbb{R})$ and $f:\mathbb{R}\to\mathbb{R}$.

1. The functional equation

$$(1.1)\quad f:\mathbb{R}\to\mathbb{R}\ :\quad f(x+y-xy)+f(xy)=f(x)+f(y)\qquad (x,y\in\mathbb{R})$$

was first presented by M. Hosszú at the International Symposium on Functional Equations held in Zakopane (Poland) in October 1967. (1.1) is referred to as the Hosszú equation. Its connection with the theory of additive functions is expressed by the following (Daróczy [3]; cf. also Blanuša [2], Davison [5], Światak [8])

Theorem 1. *Let a function* $f:\mathbb{R}\to\mathbb{R}$ *be a solution of the Hosszú equation* (1.1). *Then there exists an additive function* $A:\mathbb{R}\to\mathbb{R}$ *such that*

$$(1.2)\qquad f(x)=A(x)+f(0)$$

for all $x\in\mathbb{R}$.

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The function $A : \mathbb{R} \to \mathbb{R}$ is said to be *additive* if it satisfies the Cauchy functional equation $A(x+y) = A(x) + A(y)$ $(x, y \in \mathbb{R})$ (Aczél [1], Kuczma [7]). In the following we define a binary operation $\circ$ on $\mathbb{R}$ by

$$x \circ y := \ln\left(e^x + e^y\right) \tag{1.3}$$

for all $x, y \in \mathbb{R}$. Now we consider the functional equation

$$f : \mathbb{R} \to \mathbb{R} \ : \ f(x+y-x\circ y) + f(x \circ y) = f(x) + f(y) \quad (x, y \in \mathbb{R}). \tag{1.4}$$

Equation (1.4) is called a *Hosszú type functional equation.*

2. The extension theorem (Daróczy–Losonczi [4], cf. Kuczma [7]) plays an important role in the theory of additive functions. Now let $D \subset \mathbb{R}^2$ be an arbitrary nonvoid set. We define sets $D_1, D_2, D_3 \subset \mathbb{R}$ as follows

$$D_1 := \{x \in \mathbb{R} \mid \text{ there exists a } y \text{ such that } (x,y) \in D\},$$
$$D_2 := \{y \in \mathbb{R} \mid \text{ there exists an } x \text{ such that } (x,y) \in D\},$$
$$D_3 := \{z \in \mathbb{R} \mid z = x + y, \ (x,y) \in D\}.$$

Let $D^* := D_1 \cup D_2 \cup D_3$. We say that a function g is an *additive function* on D if $g : D^* \to \mathbb{R}$ satisfies the functional equation

$$g(x+y) = g(x) + g(y) \tag{2.1}$$

for all $(x,y) \in D$.

Theorem 2 (Extension theorem). *Let $D \subset \mathbb{R}^2$ be a nonvoid, open, and connected set. If a function g is an additive function on D then there exist a unique additive function $A : \mathbb{R} \to \mathbb{R}$ and constants $a, b \in \mathbb{R}$ such that*

$$\begin{cases} g(x) = A(x) + a & \text{for all } x \in D_1, \\ g(x) = A(x) + b & \text{for all } x \in D_2, \\ g(x) = A(x) + a + b & \text{for all } x \in D_3. \end{cases} \tag{2.2}$$

In our investigations we need the following elementary lemma.

Lemma. *Let*

$$D := \{(u,v) \mid u > 0, \ln(1 - e^{-u}) < v < -\ln(1 - e^{-u})\} \subset \mathbb{R}^2. \tag{2.3}$$

If $(u,v) \in D$ is arbitrary then there exist $x \in \mathbb{R}$ and $y > 0$ such that

$$u = x \circ y + y - \ln(e^y - 1) - (x \circ y) \circ (y - \ln(e^y - 1)), \tag{2.4}$$

$$v = x + y - x \circ y. \tag{2.5}$$

Proof. Let $(u,v) \in D$ be fixed. We define

$$x := \ln \frac{Ae^v}{A - e^v}, \qquad y := \ln A, \tag{2.6}$$

where

$$A := \left(\frac{e^u e^v}{e^u - 1} \right)^{\frac{1}{2}}. \tag{2.7}$$

Since $u > 0$, we have $A > 0$. Furthermore, $A > 1$ because the assertion is equivalent to the inequality $v > \ln(1 - e^{-u})$, thus $y > 0$. The inequality $v < -\ln(1 - e^{-u})$ implies $A > e^v$, therefore $x \in \mathbb{R}$. Substituting (2.6) and (2.7) in (2.4) and (2.5) we have that $x \in \mathbb{R}$ and $y > 0$ defined by (2.6) and (2.7) are solutions of the system of equations (2.4), (2.5). $\Diamond$

3. Now we consider the Hosszú type functional equation (1.4), where the binary operation $\circ$ is defined by (1.3).

Theorem 3. *A function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies equation* (1.4) *if and only if there exists an additive function* $A : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) = A(x) + f(0) \tag{3.1}$$

for all $x \in \mathbb{R}$.

Proof. If f has the form (3.1) then (1.4) is valid. Now suppose that $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1.4). We define a function

$$G(x, y) := f(x) + f(y) - f(x \circ y) \tag{3.2}$$

for all $x, y \in \mathbb{R}$. By the associativity of the operation $x \circ y$ $(x, y \in \mathbb{R})$, we have

$$G(x \circ y, z) + G(x, y) = G(x, y \circ z) + G(y, z) \tag{3.3}$$

for all $x, y, z \in \mathbb{R}$. We define

$$g(x) := f(x) - f(0) \qquad \text{if } x \in \mathbb{R}. \tag{3.4}$$

(3.3) and (3.2) imply

$$\begin{aligned} &g\left(x \circ y + z - (x \circ y) \circ z\right) + g(x + y - x \circ y) = \\ &= g\left(x + y \circ z - x \circ (y \circ z)\right) + g(y + z - y \circ z) \end{aligned} \tag{3.5}$$

for all $x, y, z \in \mathbb{R}$ and

$$g(0) = 0. \tag{3.6}$$

If $y > 0$ then $z := y - \ln(e^y - 1) \in \mathbb{R}$ and $y + z - y \circ z = 0$. Substituting this in (3.5), by the associativity of the operation $\circ$ and by (3.6) we have

$$\begin{gathered} g\left(x \circ y + y - \ln(e^y - 1) - (x \circ y) \circ (y - \ln(e^y - 1))\right) + \\ + g(x + y - x \circ y) = g\left(x + y \circ z - x \circ (y \circ z)\right) = \\ = g\left(x + y + y - \ln(e^y - 1) - (x \circ y) \circ (y - \ln(e^y - 1))\right). \end{gathered}$$

This equality implies

$$g(u) + g(v) = g(u + v) \tag{3.7}$$

for all $u, v \in \mathbb{R}$, with the following notations:

(3.8) $$u := x \circ y + y - \ln(e^y - 1) - (x \circ y) \circ (y - \ln(e^y - 1)),$$

(3.9) $$v := x + y - x \circ y.$$

By the lemma we know that (3.7) is true for all $(u, v) \in D$, where $D \subset \mathbb{R}^2$ is defined by (2.3), that is, g is an additive function on D. It can be easily seen that D is a nonvoid, open, and connected set in $\mathbb{R}^2$ and $D_2 = D^* = \mathbb{R}$. Th. 2 implies that there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that $g(x) = A(x) + b$ if $x \in D_2 = \mathbb{R}$, but by (3.6) $b = 0$, i.e., $g(x) = A(x)$ for all $x \in \mathbb{R}$. From (3.4) the assertion of our theorem follows.◊

4. In this section we give some corollaries of Th. 3. In the following $\mathbb{R}_+$ denotes the set of the positive real numbers.

Corollary 1. *A function $F : \mathbb{R}_+ \to \mathbb{R}$ is a solution of the functional equation*

(4.1) $$F(x+y) + F\left(\frac{xy}{x+y}\right) = F(x) + F(y) \qquad (x, y \in \mathbb{R}_+)$$

if and only if the function $L : \mathbb{R}_+ \to \mathbb{R}$ defined by

(4.2) $$L(x) := F(x) - F(1) \qquad (x \in \mathbb{R}_+)$$

satisfies the logarithm functional equation

(4.3) $$L(xy) = L(x) + L(y) \qquad (x, y \in \mathbb{R}_+).$$

Proof. If the function $L : \mathbb{R}_+ \to \mathbb{R}$ satisfies (4.3) then the function $F(x) := L(x) + F(1)$ $(x \in \mathbb{R}_+)$ is a solution of (4.1). Now suppose that $F : \mathbb{R}_+ \to \mathbb{R}$ is a solution of (4.1). We define

(4.4) $$f(t) := F(e^t) \qquad (t \in \mathbb{R}).$$

Then $f : \mathbb{R} \to \mathbb{R}$ and by (4.1) we have

$$f(t + s - t \circ s) + f(t \circ s) = F(e^{t+s-t\circ s}) + F(e^{t\circ s}) =$$
$$= F\left(\frac{e^t e^s}{e^t + e^s}\right) + F(e^t + e^s) = F(e^t) + F(e^s) = f(t) + f(s)$$

for all $t, s \in \mathbb{R}$, i.e., f is a solution of the Hosszú type functional equation (1.4). Th. 3 implies the existence of an additive function $A : \mathbb{R} \to \mathbb{R}$ for which $f(t) = A(t) + f(0)$ $(t \in \mathbb{R})$, therefore by (4.4) we have

$$F(x) = f(\ln x) = A(\ln x) + f(0) = A(\ln x) + F(1)$$

for all $x \in \mathbb{R}_+$. With the notation $L(x) := A(\ln x)$ $(x \in \mathbb{R}_+)$, equations (4.2) and (4.3) are true. This completes the proof of the corollary. ◊

Cor. 1 implies

Corollary 2. *A function $F : \mathbb{R}_+ \to \mathbb{R}$ satisfies the functional equation*

$$F(x+y) - F(x) - F(y) = F\left(\frac{1}{x} + \frac{1}{y}\right) \qquad (x, y \in \mathbb{R}_+) \tag{4.5}$$

if and only if

$$F(xy) = F(x) + F(y) \tag{4.6}$$

holds for all $x, y \in \mathbb{R}_+$.

Proof. (4.6) implies (4.5). If $F : \mathbb{R}_+ \to \mathbb{R}$ is a solution of (4.5) then with the substitution $x = y = 1$ we obtain $F(1) = 0$. Putting $x = \frac{1}{y}$ and $y = \frac{1}{x}$ in (4.5) we have

$$F\left(\frac{1}{x} + \frac{1}{y}\right) - F\left(\frac{1}{x}\right) - F\left(\frac{1}{y}\right) = F(x+y). \tag{4.7}$$

From (4.5) and (4.7)

$$F\left(\frac{1}{x}\right) + F\left(\frac{1}{y}\right) = -F(x) - F(y)$$

follows, thus, with $y = 1$,

$$F\left(\frac{1}{x}\right) = -F(x) \qquad (x \in \mathbb{R}_+). \tag{4.8}$$

By (4.8)

$$F\left(\frac{1}{x} + \frac{1}{y}\right) = F\left(\frac{x+y}{xy}\right) = -F\left(\frac{xy}{x+y}\right),$$

that is, function $F : \mathbb{R}_+ \to \mathbb{R}$ is a solution of (4.1). $f(1) = 0$ and Cor. 1 imply the assertion. ◊

Remark. Cor. 2 was also proved by Heuwers [6] using a different method.

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