Mathematica Pannonica
10/1 (1999), 67–75

SUMMABILITY ESTIMATES OF DOUBLE VILENKIN-FOURIER SERIES

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Dedicated to Professor Ferenc Schipp on his 60th birthday

Received: September 1998

MSC 1991: 42 C 10, 42 C 25, 60 G 42

Keywords: Vilenkin system, p-atoms, Hardy spaces.

Abstract: We obtain estimates of Hardy–Lorentz norms of a large class of summability methods for double Vilenkin-Fourier series which are valid for Vilenkin systems of both bounded and unbounded type. For the bounded case, these estimates contain a recent result of F. Weisz. For the unbounded case, they show that pointwise estimates are not, in general, sensitive to the order of growth of the parameters used to define the Vilenkin system. This is quite different from the uniform case.

Let $\mathbb{N}$ represent the set of natural numbers, and $\mathbf{Q} := [0,1) \times \times [0,1)$ represent the unit cube. Let $p_0, p_1, p_2, \dots$ and $q_0, q_1, q_2, \dots$ be two sequences of natural numbers with $p_n \geq 2$ and $q_n \geq 2$. For each $n \in \mathbb{N}$ set $P_n := p_0 p_1 \dots p_{n-1}$ and $Q_n := q_0 q_1 \dots q_{n-1}$, where the empty product is by definition 1. The *double Vilenkin system* associated with these parameters is the system $(w_{n,m}; n, m \in \mathbb{N})$ defined on $\mathbf{Q}$ as follows:

$$w_{n,m}(x,y) := w_n(x) w_m(y) := \prod_{k=0}^{\infty} \exp\left(\frac{2\pi i n_k x_k}{p_k}\right) \prod_{k=0}^{\infty} \exp\left(\frac{2\pi i m_k y_k}{q_k}\right),$$

where the coefficients n_k, m_k, x_k, y_k all are integers which satisfy $0 \leq \leq n_k < p_k$, $0 \leq m_k < q_k$, $0 \leq x_k < p_k$, $0 \leq y_k < q_k$, $n = \sum_{k=0}^{\infty} n_k P_k$, $m = \sum_{k=0}^{\infty} m_k Q_k$, $x = \sum_{k=0}^{\infty} x_k P_{k+1}^{-1}$, and $y = \sum_{k=0}^{\infty} y_k Q_{k+1}^{-1}$ (see Vilenkin

[5] for more details). When $p_k \equiv q_k \equiv 2$ for all k, the system $w_{n,m}$ is the double Walsh system. When $p_k = O(1)$ and $q_k = O(1)$, the system $w_{n,m}$ is called a double Vilenkin system of *bounded type.*

Let

$$\widehat{f}(k,j) := \iint_{\mathbf{Q}} f(x,y) w_{k,j}(x,y)\, d(x,y)$$

represent the *double Vilenkin–Fourier coefficients* of an $f \in L_1(\mathbf{Q})$. The *double Vilenkin–Fourier series* of f is the series whose partial sums are given by

$$S_{n,m} f := \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \widehat{f}(k,j) w_{k,j}$$

and the *Cesàro means* of the double Vilenkin–Fourier series of f are given by

$$\sigma_{n,m} f := \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \left(1 - \frac{k}{n}\right)\left(1 - \frac{j}{m}\right) \widehat{f}(k,j) w_{k,j}.$$

Cesàro summability of Vilenkin–Fourier series for systems of bounded type is fairly well understood. Building on earlier work of Schipp, Pál and Simon [3] proved for the one–dimensional case that if $f \in L^1[0,1)$ then $\sigma_n f \to f$ almost everywhere on $[0,1)$. For the two–dimensional case, Móricz, Schipp, and Wade [1] proved that if $f \in L^1(\mathbf{Q})$ and $p_n \equiv q_n \equiv 2$ for all $n \in \mathbb{N}$, then $\sigma_{P_n,Q_m} f \to f$ almost everywhere on $\mathbf{Q}$, provided $|n - m| \le \alpha$ for some $\alpha > 0$; Weisz [6] extended this result to bounded Vilenkin systems and also proved in this case that $\sigma_{n,m} f \to f$ almost everywhere on $\mathbf{Q}$, provided $\alpha^{-1} < n/m < \alpha$ for some $\alpha > 0$. None of these results extend to unbounded Vilenkin systems.

Weisz [6] proved much more than was stated in the previous paragraph. He actually obtained estimates for the Hardy–Lorentz norms of the Cesàro means of double Vilenkin–Fourier series in the bounded case. We shall generalize one of these estimates to a class of summability methods whose kernels are of the following type:

$$F_n(x) := \sum_{j=0}^{n-1} \sum_{k=0}^{p_j-1} \sum_{\ell=0}^{p_j-1} \zeta(j,k,n)\omega(j,k,\ell) D_{P_n}(x \dotplus \ell P_{j+1}^{-1}). \tag{1}$$

Notice that this includes both summability methods $S_{P_n} f$ and $\sigma_{P_n} f$ because if

$$\zeta(j,k,n) = \omega(j,k,\ell) = \begin{cases} 1 & j = k = \ell = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then F_n is the Vilenkin–Dirichlet kernel

$$D_{P_n} := \sum_{k=0}^{P_n-1} w_k,$$

and if

(2) $$\zeta(j,k,n) = \frac{kP_j}{p_jP_n} \quad\text{and}\quad \omega(j,k,\ell) = \exp(2\pi i/p_j)^{-\ell k}$$

then F_n is the Vilenkin–Fejér kernel

$$K_{P_n} := \sum_{k=0}^{P_n-1} \left(1-\frac{k}{n}\right) w_k$$

(see Pál and Simon [3]).

To state our main theorem, for each pair $0 < p,q < \infty$, let $H_{p,q}(\mathbf{Q})$ represent the Vilenkin Hardy–Lorentz spaces introduced by Weisz [6] using Vilenkin martingales and quadratic variation instead of distributions and maximal operators. (These spaces reduce to the usual Vilenkin Hardy spaces $H_p(\mathbf{Q})$ when $p = q$.) Also let $\dot{+}$ and $\dot{-}$ represent addition and subtraction (respectively) which is inherited from the underlying Vilenkin group, e.g., if $x = \sum_{k=0}^{\infty} x_k P_{k+1}^{-1}$, and $y = \sum_{k=0}^{\infty} y_k P_{k+1}^{-1}$, then $x \dot{+} y := \sum_{k=0}^{\infty} (x_k \oplus y_k) P_{k+1}^{-1}$, where $\oplus$ denotes addition modulo p_k—see Vilenkin [5] for details.

We shall prove the following result.

Theorem 1. *Let $\alpha > 0$, $0 < \delta \le 1$, $r \ge 1$, F_n be a kernel of type* (1), *and*

(3) $$(\mathcal{F}_{n,m}f)(x,y) := \int_0^1 \int_0^1 f(t,u) F_n(x \dot{-} t) F_m(y \dot{-} u)\, dt\, du.$$

Suppose $\beta(n,m,r)$ are positive numbers which satisfy

(4) $$\frac{P_K Q_K}{P_{K-r} Q_{K-r}} \sum_{j=0}^{K-r-1} p_j \left(\sum_{k=0}^{p_j-1} \frac{|\zeta(j,k,n)|\, P_n}{\beta(n,m,r) P_K} \|F_m\|_1 \right)^{\delta} = O(1)$$

and

$$\frac{P_K Q_K}{P_{K-r} Q_{K-r}} \sum_{j=0}^{K-r-1} q_j \left(\sum_{k=0}^{q_j-1} \frac{|\zeta(j,k,m)|\, Q_m}{\beta(n,m,r) Q_K} \|F_n\|_1 \right)^{\delta} = O(1),$$

uniformly in n and m, as $K \to \infty$. If the $\omega(j,k,\ell)$'s are bounded and

(5) $$\|F_n\|_1 \, \|F_m\|_1 = O(\beta(n,m,r))$$

as $n, m \to \infty$, then

$$\mathcal{F}_\alpha^*(f) := \sup_{|m-n| \le \alpha} \frac{|\mathcal{F}_{n,m}(f)|}{\beta(n,m,r)}$$

is of weak type $(1,1)$ and for every $\delta < p \le 1$ and $0 < q \le \infty$ there is a *constant* $C_{p,q}$ *which depends only on* p *and* q *such that*

$$||\mathcal{F}_\alpha^* f||_{p,q} \le C_{p,q}||f||_{H_{p,q}(\mathbf{Q})}.$$

Proof. We begin the proof with additional notation and terminology. A *Vilenkin rectangle* of order n, m is a rectangle of the form $R := [kP_n^{-1}, (k+ +1)P_n^{-1}) \times [\ell Q_m^{-1}, (\ell+1)Q_m^{-1})$, where k and ℓ are integers which satisfy $0 \le k < P_n$, $0 \le \ell < Q_m$. The $\sigma-$ algebra generated by all Vilenkin rectangles of order n, m will be denoted by $\mathcal{A}_{n,m}$. Given a rectangle $R \in \mathcal{A}_{n,m}$ and $r \in \mathbb{N}$, the *r–fold expansion* of R will be the Vilenkin rectangle $R^r \in \mathcal{A}_{n-r,m-r}$ which satisfies $R \subset R^r$. A *Vilenkin martingale* is a sequence of integrable functions $(f_{n,m}; n, m \in \mathbb{N})$ such that each $f_{n,m}$ is $\mathcal{A}_{n,m}$ measurable, and the conditional expectation operator with respect to these σ–algebras satisfies $E_{k,\ell}(f_{n,m}) = f_{k,\ell}$ for $k \le n$ and $\ell \le m$. It is well–known that the partial sums $S_{P_n,Q_m}f$ of the double Vilenkin–Fourier series of any $f \in L_1(\mathbf{Q})$ is a Vilenkin martingale.

Let $0 < p < \infty$. A *p–atom* on $\mathbf{Q}$ is a bounded measurable function a on $\mathbf{Q}$ which satisfies

i) a is of *mean zero*, i.e.,

$$\iint_{\mathbf{Q}} a(x,y)d(x,y) = 0,$$

ii) there is a Vilenkin rectangle $R \in \mathcal{A}_{K,K}$ such that $||a||_\infty \le |R|^{-1/p}$,

iii) a is supported on R.

The only thing we need to know about Hardy–Lorentz spaces is the following extrapolation theorem (see Weisz [6], Th. 2):

Lemma. *Suppose* T *is a sublinear operator defined on Vilenkin martingales and* $0 < \delta \le 1$. *If for every* $\delta < p \le 1$ *there exists an* $r \in \mathbb{N}$ *and a constant* $C_{p,r}$, *which depends only on* p *and* r *such that*

$$\iint_{\mathbf{Q}\setminus R^r} |Ta|^p \, d(x,y) \le C_{p,r}$$

holds for all p–atoms a, *and if* T *is bounded from* $L_\infty(\mathbf{Q})$ *to* $L_\infty(\mathbf{Q})$, *then* T *is of weak type (1,1) on* $L_1(\mathbf{Q})$ *and for each* $\delta < p \le 1$, $0 < q \le \infty$ *there is a constant* $C_{p,q}$ *which depends only on* p *and* q *such that*

$$||Tf||_{p,q} \le C_{p,q}||f||_{H_{p,q}(\mathbf{Q})}.$$

We are now prepared to prove the theorem. Let r be the integer which satisfies $r - 1 < \alpha \le r$. By hypothesis (5), $\mathcal{F}_\alpha^*$ is of type (∞, ∞). Hence by Weisz's Lemma, it suffices to show $\mathcal{F}_\alpha^*$ is p–quasilocal for all $\delta < p \le 1$, i.e., for each such p there is a constant $C_{p,r}$ such that if a is a p–atom supported on some "square" $R := I \times J$, then

$$(6) \qquad \iint_{\mathbf{Q}\setminus R^r} |(\mathcal{F}^*_\alpha a)(x,y)|^p \, d(x,y) \le C_{p,r}.$$

Since R is a square, choose $K \in \mathbb{N}$ such that $|I| = P_K^{-1}$ and $|J| = Q_K^{-1}$. If $n < K$ and $m < K$, then both w_n and w_m are constant on I and J. Since a is of mean zero, it follows that $\widehat{a}(n,m) = 0$. Thus we may suppose that either n or m is $\ge K$. If $n \ge K$, then the conditions $|n-m| \le \alpha$ and $r-1 < \alpha \le r$ imply $m \ge n-\alpha \ge n-r \ge K-r$. Hence we may assume that both m and n are $\ge K-r$.

To prove (6), notice first that $\mathbf{Q}\setminus R^r$ can be broken into 8 pieces: two horizontal pieces $([0,1)\setminus I^r)\times J^r$; two vertical pieces $I^r \times ([0,1)\setminus J^r)$; and four rectangular pieces $([0,1)\setminus I^r)\times([0,1)\setminus J^r)$. Estimates of $\mathcal{F}^*_\alpha$ over the horizontal and vertical pieces are similar, and the estimates over the remaining rectangular pieces are simpler. Consequently, we shall supply the details for the horizontal estimates only.

Fix $(x,y) \in ([0,1)\setminus I^r)\times J^r$. Since a is supported on $R \equiv I \times J$ and $|a| \le |R|^{-1/p}$, it follows from (3) that

$$|(\mathcal{F}_{n,m}a)(x,y)| \le \int_0^1 \int_I |a(t,u)|\,|F_n(x \overset{\bullet}{-} t)|\,|F_m(y \overset{\bullet}{-} u)|\,dt\,du \le$$
$$\le |R|^{-1/p}\|F_m\|_1 \int_I |F_n(x \overset{\bullet}{-} t)|\,dt.$$

Using (1) and the fact that the $\omega(j,k,\ell)$'s are bounded, we can continue this estimate as follows.

$$|(\mathcal{F}_{n,m}a)(x,y)| \le$$
$$\le C|R|^{-1/p}\|F_m\|_1 \sum_{j=0}^{n-1}\sum_{k=0}^{p_j-1}\sum_{\ell=0}^{p_j-1} |\zeta(j,k,n)| \int_I |D_{P_n}(x \overset{\bullet}{+} \ell P_{j+1}^{-1} \overset{\bullet}{-} t)|\,dt.$$

(Here, and elsewhere, C denotes an absolute constant which may change from line to line.)

Recall that

$$(7) \qquad D_{P_n}(x) \equiv P_n \chi_{[0,P_n^{-1})}(x) := \begin{cases} P_n & 0 \le x < P_n^{-1} \\ 0 & P_n^{-1} \le x < 1. \end{cases}$$

Since $x \notin I^r$ and D_{P_n} is supported on $[0, P_n^{-1})$, it follows that if $n \ge$ $\ge j \ge K-r$ then both $D_{P_n}(x \overset{\bullet}{-} t)$ and $D_{P_n}(x \overset{\bullet}{+} \ell P_{j+1}^{-1} \overset{\bullet}{-} t)$ are identically zero for $t \in I$ and $0 \le \ell < p_j$ (because under these conditions, x and $x \overset{\bullet}{+} \ell P_{j+1}^{-1}$ both lie outside I^r so the coefficients of t cannot cancel all "lower order" the coefficients of x or $x \overset{\bullet}{+} \ell P_{j+1}^{-1}$). Thus

$$|(\mathcal{F}_{n,m}a)(x,y)| \le C|R|^{-1/p}\|F_m\|_1 \cdot$$
$$\cdot \sum_{j=0}^{K-r-1}\sum_{k=0}^{p_j-1}\sum_{\ell=0}^{p_j-1}|\zeta(j,k,n)|\int_I D_{P_n}(x \dot{+} \ell P_{j+1}^{-1} \dot{-} t)\,dt. \tag{8}$$

Since Lebesgue measure is translation invariant with respect to $\dot{+}$, we may suppose that $I=[0,P_K^{-1})$. Since $n \ge K-r$, and $D_{P_i} \equiv P_i\chi_{[0,P_i^{-1})}$ for all $i \in \mathbb{N}$, we have both $\frac{1}{P_n}D_{P_n} \le \frac{1}{P_{K-r}}D_{P_{K-r}}$ and

$$\int_I D_{P_i}(x \dot{+} \ell P_{j+1}^{-1} \dot{-} t)\,dt = \frac{P_i}{P_K}\chi_{[\xi_{j\ell},\xi_{j\ell}+P_i^{-1})}(x)$$

for $j < i \le K-1$ and $x \notin I^r$, where $\xi_{j\ell} := (p_j-\ell)P_{j+1}^{-1}$. Combining these observations with (8), we obtain

$$|(\mathcal{F}_{n,m}a)(x,y)| \le C|R|^{-1/p}\|F_m\|_1 \cdot$$
$$\cdot \sum_{j=0}^{K-r-1}\sum_{\ell=0}^{p_j-1}\sum_{k=0}^{p_j-1}|\zeta(j,k,n)|\frac{P_n}{P_{K-r}}\frac{P_{K-r}}{P_K}\chi_{[\xi_{j\ell},\xi_{j\ell}+P_{K-r}^{-1})}(x).$$

Cancelling the P_{K-r}'s and using the inequality $(a+b)^p \le a^p + b^p$ (valid for all $a,b \ge 0$ and $0<p\le 1$), it follows that

$$|(\mathcal{F}_{n,m}a)(x,y)|^p \le C^p|R|^{-1}\|F_m\|_1^p \cdot$$
$$\cdot \sum_{j=0}^{K-r-1}\sum_{\ell=0}^{p_j-1}\left(\sum_{k=0}^{p_j-1}|\zeta(j,k,n)|\frac{P_n}{P_K}\right)^p \chi_{[\xi_{j\ell},\xi_{j\ell}+P_{K-r}^{-1})}(x)$$

for all $x \in [0,1)\setminus I^r$. Since $|R| = P_K^{-1}Q_K^{-1}$, we arrive at the penultimate estimate

$$\left|\frac{(\mathcal{F}_{n,m}a)(x,y)}{\beta(n,m,r)}\right|^p \le C^p P_K Q_K\|F_m\|_1^p \cdot$$
$$\cdot \sum_{j=0}^{K-r-1}\sum_{\ell=0}^{p_j-1}\left(\sum_{k=0}^{p_j-1}\frac{|\zeta(j,k,n)|}{\beta(n,m,r)}\frac{P_n}{P_K}\right)^p \chi_{[\xi_{j\ell},\xi_{j\ell}+P_{K-r}^{-1})}(x) \tag{9}$$

for all $x \in [0,1)\setminus I^r$. Since $|J^r| = Q_{K-r}^{-1}$ and $|[\xi_{j\ell},\xi_{j\ell}+P_{K-r}^{-1})| = P_{K-r}^{-1}$ for each j and ℓ, we conclude that

$$\int_{[0,1)\setminus I^r}\int_{J^r}|(\mathcal{F}_\alpha^* a)(x,y)|^p\,dy\,dx \le C^p \sup_{m,n,K}\frac{P_KQ_K}{P_{K-r}Q_{K-r}} \cdot$$
$$\cdot \sum_{j=0}^{K-r-1} p_j\left(\sum_{k=0}^{p_j-1}\frac{|\zeta(j,k,n)|\,P_n}{\beta(n,m,r)P_K}\|F_m\|_1\right)^p$$

which is bounded by hypothesis (4) since $p > \delta$.◇

By choosing different ζ's, ω's, and β's, we can use this theorem to generate many results about a variety of summability methods including (C,α) methods and methods involving subsequences of partial sums of Sf. To demonstrate how this can be done, to verify that the hypotheses of the theorem are not vacuous, and to illustrate that the theorem contains new information even in the Cesàro case, we shall prove two results about the Cesàro means $\sigma_{P_n,Q_m}f$. The first result is Th. 4ii) in Weisz [6].

Corollary 1. *Let $\alpha > 0$ and $0 < \delta \le 1$. Suppose the double Vilenkin system is of bounded type, i.e., $p_n = O(1)$ and $q_m = O(1)$ as $n,m \to \infty$. Then*

$$\sigma^*_\alpha f := \sup_{|m-n|\le\alpha} |\sigma_{P_n,Q_m}(f)|$$

is of weak type $(1,1)$ and for every $\delta < p \le 1$ and $0 < q \le \infty$ there is a constant $C_{p,q}$ which depends only on p and q such that

$$\|\sigma^*_\alpha f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}(\mathbf{Q})}.$$

Proof. Set $\beta(n,m,r) \equiv 1$ and suppose (2) holds. Since $\|K_m\|_1 = O(1)$ as $m \to \infty$ (see Pál and Simon [3]) and r is fixed, it is easy to check for $F_m \equiv K_{Q_m}$ that

$$\frac{P_K Q_K}{P_{K-r}Q_{K-r}} \sum_{j=0}^{K-r-1} p_j \left(\sum_{k=0}^{p_j-1} \frac{|\zeta(j,k,n)|\, P_n}{\beta(n,m,r)P_K}\|F_m\|_1\right)^\delta \le$$

$$\le C\sum_{j=0}^{K-r-1} p_j \left(\sum_{k=0}^{p_j-1} \frac{kP_j}{p_jP_n}\frac{P_n}{P_K}\right)^\delta \le$$

$$\le C\sum_{j=0}^{K-r-1}\left(\frac{P_j}{P_K}\right)^\delta \le C\sum_{j=0}^{K-r-1}\left(\frac{1}{2^{K-j}}\right)^\delta < \infty$$

since $P_j = p_0 \dots p_{j-1}$, $2 \le p_i \le M$ for all i and some finite M, and $0 < \delta \le 1$. Thus the proof is complete by applying Th. 1.◊

Th. 1 can also used to obtain results for Vilenkin systems of unbounded type.

Corollary 2. *Let $\alpha > 0$ and $0 < \delta \le 1$. Suppose $p_n \uparrow \infty$ and $q_m \uparrow \infty$ as $n,m \to \infty$. If $\beta(n,m,r) := \max\{p_{n-1}q_{m-1}\}^{1/\delta} \cdot \max\{p_{n+r-1}q_{m+r-1}\}^{2r/\delta}$, then*

$$\sigma^*_\alpha f := \sup_{|m-n|\le\alpha} \frac{|\sigma_{P_n,Q_m}(f)|}{\beta(n,m,r)}$$

is of weak type $(1,1)$ and for every $\delta < p \le 1$ and $0 < q \le \infty$ there is a constant $C_{p,q}$ which depends only on p and q such that

$$\|\sigma_\alpha^* f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}(\mathbf{Q})}.$$

Proof. Using the "q_m" version of (2) and the translation invariance of Lebesgue measure with respect to $\dot{+}$ (see Vilenkin [5]), it is clear that

$$\|K_{Q_m}\|_1 \le \sum_{j=0}^{m-1} \frac{Q_j}{Q_m} q_j \sum_{\ell=0}^{q_j-1} \int_0^1 D_{Q_m}(x \dot{+} \ell Q_{j+1}^{-1})\, dx \le \sum_{j=0}^{m-1} \frac{Q_j}{Q_m} q_j^2 =$$

$$= q_{m-1} + \sum_{j=0}^{m-2} \frac{q_j}{q_{j+1}\cdots q_{m-1}} \le$$

$$\le q_{m-1} + \sum_{j=0}^{m-2} \frac{q_{j+1}}{q_{j+1}\cdots q_{m-1}} \le q_{m-1} + \sum_{j=1}^{\infty} \frac{1}{2^j} = O(q_{m-1})$$

as $m \to \infty$. Moreover, since the p's and q's are increasing, it is also true that

$$q_{m-1}^\delta p_j = q_{m-1}^\delta p_j^{1-\delta+\delta} \le$$

$$\le \max\{p_{n-1}, q_{m-1}\}^\delta \max\{p_{n-1}, q_{m-1}\}^{1-\delta} p_j^\delta = \max\{p_{n-1}, q_{m-1}\} p_j^\delta$$

for all $0 \le j < K - r$ and

$$\frac{P_K}{P_{K-r}} \frac{Q_K}{Q_{K-r}} = p_{K-r}\cdots p_{K-1} \cdot q_{K-r}\cdots q_{K-1} \le \max\{p_{n+r-1}, q_{m+r-1}\}^{2r},$$

i.e.,

$$\frac{P_K Q_K}{P_{K-r} Q_{K-r}} \left(\frac{q_{m-1}}{p_j \beta(n,m,r)}\right)^\delta \le \frac{1}{p_j}.$$

Since $F_m \equiv K_{Q_m}$, it follows that

$$\frac{P_K Q_K}{P_{K-r} Q_{K-r}} \sum_{j=0}^{K-r-1} p_j \left(\sum_{k=0}^{p_j-1} \frac{|\zeta(j,k,n)|\, P_n}{\beta(n,m,r) P_K} \|F_m\|_1\right)^\delta \le$$

$$\le C \sum_{j=0}^{K-r-1} p_j \left(\sum_{k=0}^{p_j-1} \frac{k P_j}{p_j P_n} \frac{P_n}{P_K} \frac{q_{m-1}}{\beta(n,m,r)}\right)^\delta \le$$

$$\le C \sum_{j=0}^{K-r-1} \left(\frac{p_j^2 P_j}{P_K}\right)^\delta \equiv C \sum_{j=0}^{K-r-1} \left(\frac{p_j}{p_{j+1}\cdots p_{K-1}}\right)^\delta \le C \sum_{j=0}^{\infty} \left(\frac{1}{2^j}\right)^\delta < \infty. \Diamond$$

For uniform convergence, Simon [4] proved that estimates of Vilenkin–Fourier series are sensitive to the rate of growth of the parameters p_n and q_m. We see that this is NOT the case for pointwise estimates since

Cor. 2 holds for parameters which grow arbitrarily slowly and arbitrarily quickly.

Th. 1 also contains new information about growth of Cesàro means for unbounded Vilenkin systems even in the one–dimensional case. For example, the same techniques show that if $p_n \uparrow \infty$ then $|\sigma_{P_n} f(x) - f(x)| = = o(P_n^{1/n})$ as $n \to \infty$.

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