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A PALEY TYPE INEQUALITY FOR TWO-PARAMETER VILENKIN–FOURIER COEFFICIENTS

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Abstract: In our earlier paper Simon and Weisz [5] we gave an extension for H^p $(0 < p \leq 1)$ spaces of a Paley type inequality wit respect to the Vilenkin-Fourier coefficients. In the present work we formulate a two-parameter version of this result. By duality a Kintchin type inequality follows in the (bounded) two-parameter case.

1. Introduction

The classical inequality due to Paley [2] is well-known in the harmonic analysis. Namely, the Walsh-Fourier coefficients of a function $f \in L^p$ $(1 < p)$ satisfy the condition $\sum_{k=0}^{\infty} |\hat{f}(2^k)|^2 < \infty$. The analogous statement fails to hold for $p = 1$. However, if we replace L^1 here by the (dyadic) Hardy space H^1 then the sum in the question will be finite. The same conclusion holds also in the two-parameter case (see Coifman and Weiss [1]). In Simon and Weisz [5] we extended Paley's result for

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H^p $(0 < p \leq 1)$ spaces taking Vilenkin-Fourier coefficients. (In this connection see also Weisz [9].) In the present paper a two-parameter version of Simon and Weisz [5] will be investigated.

2. Preliminaries and notations

In this section the most important definitions and notations with respect to the two-parameter Vilenkin systems will be introduced. In this connection we refer to Vilenkin [7], Simon and Weisz [6] and to the books written by Schipp, Wade, Simon, Pál [3] and Weisz [8].

Let $m = (m_0, m_1, ..., m_k, ...)$ be a sequence of natural numbers with terms m_k greater than 1 $(k \in \mathbb{N} := \{0, 1, ...\})$ and for all $k \in \mathbb{N}$ denote Z_{m_k} the m_k-th discrete cyclic group represented by $\{0, 1, ..., m_k - 1\}$. Furthermore, let G_m be the complete direct product of Z_{m_k}'s. Then G_m forms a compact Abelian group with Haar measure 1. The elements of G_m are sequences of the form $(x_0, x_1, ..., x_k, ...)$, where $x_k \in Z_{m_k}$ for every $k \in \mathbb{N}$.

We define the *intervals* in G_m as follows. First of all let

$$I_n(0) := \{(x_0, x_1, ..., x_k, ...) \in G_m : x_j = 0 \qquad (j = 0, ..., n-1)\}$$

$$(0 \neq n \in \mathbb{N}, I_0(0) := G_m),\ I_n(x) := x + I_n(0) \ \ (n \in \mathbb{N}) \text{ and}$$

$$I_n(x, k) := \{(y_0, y_1, ...) \in I_n(x) : y_n = k\} \qquad (x \in G_m, k \in Z_{m_n}).$$

If $n \in \mathbb{N}, x \in G_m$ and $\mathcal{U}$ is a *dyadic subset* of Z_{m_n} (for more details see Simon [4] and Simon and Weisz [5]) then the set $I = \cup_{k \in \mathcal{U}} I_n(x, k)$ is called *interval*. Especially, $I_n(x)$ is also interval which will be called simple interval.

Let $G := G_m \times G_m$ be the cartesian product of G_m's then G is also a compact Abelian group. If $I, J \subset G_m$ are intervals and $|I| = |J|$ (where $|I|$ and $|J|$ is the measure of I and J, resp.) then $I \times J$ is called *m-adic square*. In the special case $I = I_n(x), J = I_n(y)$ $(x, y \in G_m, n \in \mathbb{N})$ $I \times J$ is a so-called *simple m-adic square.*

By means of m-adic squares we define a sequence $\mathcal{F}_{j,u}^{l,v}$ $(j, u, l, v \in \in \mathbb{N}, l < [\log_2 m_j], v < [\log_2 m_u])$ of σ-algebras as in Simon and Weisz [6]. The concept of martingales $f = (f_{j,u}^{l,v})$ with respect to this sequence will be taken in the usual way (see Simon and Weisz [6]).

Denote the conditional expectation operator relative to $\mathcal{F}_{j,u}^{l,v}$ by $E_{j,u}^{l,v}$ and let $f^* := \sup_{j,l} |f_{j,j}^{l,l}|$ and $\sigma(f) := \left(\sum_{n=0}^{\infty} E_{n-1,n-1}^{0,0} |f_{n,n} - f_{n-1,n-1}|^2\right)^{1/2}$ be the diagonal maximal function and the conditional quadratic variation of f, resp. Let $0 < p < \infty$ be given and define the *Hardy spaces* $H^p(G), H^p_\sigma(G)$ as the sets of martingales f for which $\|f\|_{H^p} := \|f^*\|_p < \infty$ and $\|f\|_{H^p_\sigma} := \|\sigma(f)\|_p < \infty$, resp.

We shall need the atomic characterization of $H^p(G)$, H^p_σ $(0 < p \leq \leq 1)$. For this purpose we introduce the concept of atoms. Namely, a function $a \in L^2(G)$ is called a *p-atom* if supp $a \subset I \times J$ for an m-adic square $I \times J$, $\|a\|_2 \leq |I \times J|^{1/2-1/p} = |I|^{1-2/p}$ and $\int_{I\times J} a = 0$. If $I \times J$ is a simple m-adic square then a will be called a *simple p-atom.* Hence, the atomic characterization of $H^p(G), H^p_\sigma(G)$ reads as follows.

Theorem 1 [8, Weisz]. *A martingale $f = (f^{l,l}_{j,j}; (j,l) \in \mathbb{N}^2, l \leq [\log_2 m_j]-1)$ is in $H^p(G)$ $(0 < p \leq 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that $\sum_{k=0}^{\infty} |\mu_k|^p < \infty$ and*

$$\sum_{k=0}^{\infty} \mu_k E^{l,l}_{j,j} a^k = f^{l,l}_{j,j} \tag{1}$$

for all $j, l \in \mathbb{N}, l \leq [\log_2 m_j] - 1$. Moreover, $\|f\|_{H^p} \sim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decompositions of the form (1). If we replace $H^p(G)$ by $H^p_\sigma(G)$ and the p-atoms by simple p-atoms, then the corresponding theorem holds with the restriction $l = 0$.

It is well-known (see e.g. Weisz [8]) that the dual of $H^1(G)$ is the BMO(G) space, i.e. the space of all functions $f \in L^2(G)$ for which

$$\|f\|_{BMO} := \sup_{I\times J} \left(|I \times J|^{-1} \int_{I\times J} |f - |I \times J|^{-1} \int_{I\times J} f|^2\right)^{1/2} < \infty,$$

where the supremum is taken over all m-adic squares. If we take only simple m-adic squares here then we get the $\mathcal{B}MO(G)$ space, which is the dual of $H^1_\sigma(G)$ (see Weisz [8]).

The characters of G_m (the so-called *Vilenkin system*) form a complete orthonormal system in $L^1(G_m)$. For the description of this system let $r_n(x) := \exp \frac{2\pi i x_n}{m_n}$ $(n \in \mathbb{N}, x = (x_0, x_1, ...) \in G_m, i := \sqrt{-1})$. Then the r_n's and their finite products are evidently characters. If we write each $n \in \mathbb{N}$ uniquely in the form (called m-adic decomposition of n) $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}$ $(k \in \mathbb{N})$ then the characters of G_m are the functions $\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k}$.

A good property of the kernels $D_{M_n} := \sum_{k=0}^{M_n-1} \Psi_k$ $(n \in \mathbb{N})$ will be frequently used. Namely,

$$(2) \qquad D_{M_n}(x) = \begin{cases} M_n & (x \in I_n(0)) \\ 0 & (x \in G_m \setminus I_n(0)), \end{cases}$$

where $M_0 := 1, M_{k+1} := m_0 \cdot \ldots \cdot m_k \quad (k \in \mathbb{N})$. In the special case $m_k = 2$ $(k \in \mathbb{N})$ we get the classical Walsh-Paley system.

The two-parameter Vilenkin system is defined as the Kronecker products of the Vilenkin functions, i.e. for $(j,k) \in \mathbb{N}^2$ let $\Psi_{j,k}(x,y) : := \Psi_j(x)\Psi_k(y)$ $((x,y) \in G)$. The Fourier coefficients of a function $f \in \in L^1(G)$ with respect to the system $(\Psi_{j,k})$ are denoted by $\hat{f}(j,k)$, i.e. $\hat{f}(j,k) := \int_G f\overline{\Psi}_{j,k}$ $((j,k) \in \mathbb{N}^2)$. (The bar stands for complex conjugation.) This definition can be extended to martingales in a usual way (see Weisz [8]).

Throughout this paper $C_p, C_\beta, \ldots$ will denote positive constants depending only on $p, \beta, \ldots$, not always the same in different occurences.

3. Results

The sequence m will be called *power-like* if there exists a constant $q \geq 1$ such that for all $n \in \mathbb{N}$ the inequality $m_{n+1} \leq q m_n$ holds. Thus all bounded m's are power-like and for example the unbounded m with $m_n := n+2 \quad (n \in \mathbb{N})$ is also such a sequence. However, if $m_n := 2$ for even n and $m_n := n+2$ if n is odd then m is trivially not power-like. We shall use the next notation: if $0 < \alpha \leq 1 \leq \beta$ are given then $\sum_\alpha^\beta$ means a summation with respect to the indices $k, l \in \mathbb{N}$ for which $\alpha \leq M_k/M_l \leq \beta$ holds.

Then our main theorem is

Theorem 2. *Assume that m is power-like and $0 < p \leq 1, 0 < \alpha \leq 1 \leq \beta$ are given. Then there exists a constant C depending only on p, m, α, β such that the inequality*

$$\Big(\sum\nolimits_\alpha^\beta (m_k m_l)^{1-2/p}(M_k M_l)^{2-2/p} \sum_{j=1}^{m_k-1}\sum_{n=1}^{m_l-1} |\hat{f}(jM_k, nM_l)|^2\Big)^{1/2} \leq C\|f\|_{H^p}$$

holds for all $f \in H^p(G)$.

Proof. Taking into account Th. 1 it is enough to show that

$$(3) \quad \sup \sum\nolimits_\alpha^\beta (m_k m_l)^{1-2/p}(M_k M_l)^{2-2/p} \cdot \sum_{j=1}^{m_k-1}\sum_{n=1}^{m_l-1} |\hat{a}(jM_k, nM_l)|^2 < \infty,$$

where the supremum is taken over all p-atoms a. That is, let a be a p-atom with support $I \times J$, where I, J are intervals, $|I| = |J| = \gamma/M_{N+1}$ for some $N \in \mathbb{N}$ and $\gamma = 2, ..., m_N$, $\int_{I\times J} a = 0$ and $\|a\|_2 \le |I \times J|^{1/2-1/p} = \gamma^{1-2/p} M_{N+1}^{2/p-1}$ (see the definition of p-atoms). Then $\hat{a}(u, v) = 0$ if $u, v = 0, ..., M_N - 1$. Therefore $\hat{a}(jM_k, nM_l) = 0$ for all $k, l = 0, ..., N-1$ and $j = 1, ..., m_k - 1, n = 1, ..., m_l - 1$. This means that $k \ge N$ or $l \ge N$ can be assumed in (3).

Decompose the sum $\sum_\alpha^\beta$ in (3) in the following way:

$$\sum\nolimits_\alpha^\beta = \sum_{k=l=N}^{\infty} + \sum_{k=N}^{\infty} \sum_{\substack{l=0 \\ l<k \\ M_k \le \beta M_l}}^{\infty} + \sum_{l=N}^{\infty} \sum_{\substack{k=0 \\ k<l \\ \alpha M_l \le M_k}}^{\infty} =: $$
$$=: \sum\nolimits^{(1)} + \sum\nolimits^{(2)} + \sum\nolimits^{(3)}.$$

First we investigate the sum

$$\sum\nolimits^{(1)} = m_N^{2-4/p} M_N^{4-4/p} \sum_{j,n=1}^{m_N - 1} |\hat{a}(jM_N, nM_N)|^2 +$$
$$+ \sum_{k=N+1}^{\infty} m_k^{2-4/p} M_k^{4-4/p} \sum_{j,n=1}^{m_k-1} |\hat{a}(jM_k, nM_k)|^2 =: \sum\nolimits^{(11)} + \sum\nolimits^{(12)}.$$

Since

$$|\hat{a}(jM_N, nM_N)| = |\hat{a}(jM_N + u, nM_N + v)| \qquad (u, v = 0, ..., M_N - 1)$$

we have

$$\sum\nolimits^{(11)} = m_N^{2-4/p} M_N^{4-4/p} M_N^{-2} \sum_{j,n=1}^{m_N-1} \sum_{u,v=0}^{M_N-1} |\hat{a}(jM_N + u, nM_N + v)|^2 \le$$
$$\le m_N^{2-4/p} M_N^{2-4/p} M_N^{-2} \|a\|_2^2 \le M_{N+1}^{2-4/p} \gamma^{2-4/p} M_{N+1}^{4/p-2} \le 1.$$

Denote $\mathcal{U}_I, \mathcal{U}_J$ dyadic subsets of Z_{m_N} such that

$$I = \bigcup_{u \in \mathcal{U}_I} I_N(x_I, u) \ , \ J = \bigcup_{v \in \mathcal{U}_J} I_N(x_J, v)$$

for some $x_I, x_J \in G_m$. Furthermore, define $a_{\nu,\mu}$ $(\nu \in \mathcal{U}_I, \mu \in \mathcal{U}_J)$ by

$$a_{\nu,\mu}(\tau, t) := \begin{cases} a(\tau, t) & (\tau \in I_N(x_I, \nu), t \in I_N(x_J, \mu)) \\ 0 & ((\tau, t) \in G^2 \setminus (I_N(x_I, \nu) \times I_N(x_J, \mu)). \end{cases}$$

Then for all $N + 1 \le k \in \mathbb{N}, j, n = 1, ..., m_k - 1$ and for all $u, v = 0, ..., M_{N+1} - 1$ we obtain

$$|\hat{a}(jM_k, nM_k)|^2 = |\sum_{\substack{\nu\in\mathcal{U}_I\\ \mu\in\mathcal{U}_J}} \hat{a}_{\nu,\mu}(jM_k, nM_k)|^2 \le \gamma^2 \sum_{\substack{\nu\in\mathcal{U}_I\\ \mu\in\mathcal{U}_J}} |\hat{a}_{\nu,\mu}(jM_k, nM_k)|^2 =$$

$$= \gamma^2 M_{N+1}^{-2} \sum_{\substack{\nu\in\mathcal{U}_I\\ \mu\in\mathcal{U}_J}} \sum_{u,v=0}^{M_{N+1}-1} |\hat{a}_{\nu,\mu}(jM_k + u, nM_k + v)|^2.$$

This implies

$$\sum\nolimits^{(12)} \le \gamma^2 M_{N+1}^{-2} \sum_{k=N+1}^{\infty} m_k^{2-4/p} M_k^{4-4/p} \sum_{j,n=1}^{m_k-1} \times$$

$$\times \sum_{\substack{\nu\in\mathcal{U}_I\\ \mu\in\mathcal{U}_J}} \sum_{u,v=0}^{M_{N+1}-1} |\hat{a}_{\nu,\mu}(jM_k + u, nM_k + v)|^2 \le$$

$$\le \gamma^2 M_{N+1}^{-2} M_{N+1}^{4-4/p} \sum_{k=N+1}^{\infty} \sum_{j,n=1}^{m_k-1} \sum_{\substack{\nu\in\mathcal{U}_I\\ \mu\in\mathcal{U}_J}} \sum_{u,v=0}^{M_{N+1}-1} |\hat{a}_{\nu,\mu}(jM_k + u, nM_k + v)|^2 \le$$

$$\le \gamma^2 M_{N+1}^{2-4/p} \|a\|_2^2 \le \gamma^{4-4/p} \le 1.$$

Now let the sum $\sum^{(2)}$ be investigated in the following way:

$$\sum\nolimits^{(2)} = m_N^{1-\frac{p}{2}} M_N^{2-\frac{2}{p}} \sum_{\substack{l=0\\ M_N\le\beta M_l}}^{N-1} m_l^{1-\frac{2}{p}} M_l^{2-\frac{2}{p}} \sum_{j=1}^{m_N-1} \sum_{n=1}^{m_l-1} |\hat{a}(jM_N, nM_l)|^2 +$$

$$+ \sum_{k=N+1}^{\infty} \sum_{\substack{l=0\\ M_k\le\beta M_l}}^{N-1} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\hat{a}(jM_k, nM_l)|^2 +$$

$$+ \sum_{k=N+1}^{k-1} \sum_{\substack{l=N+1\\ M_k\le\beta M_l}}^{\infty} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\hat{a}(jM_k, nM_l)|^2 =:$$

$$=: \sum\nolimits^{(21)} + \sum\nolimits^{(22)} + \sum\nolimits^{(23)}.$$

Recall that $|\hat{a}(jM_N, nM_l)| = |\hat{a}(jM_N, 0)| = |\hat{a}(jM_N+u, v)|$ for all $N-1 \ge$ $\ge l \in \mathbb{N}, n = 1, ..., m_l - 1$ and for all $u, v = 0, ..., M_N - 1$. Therefore it follows that

$$\sum^{(21)} = m_N^{1-p/2} M_N^{2-2/p} \sum_{\substack{l=0\\ M_N \le \beta M_l}}^{N-1} m_l^{1-2/p} M_l^{2-2/p} (m_l - 1) M_N^{-2} \times$$

$$\times \sum_{j=1}^{m_N-1} \sum_{u,v=0}^{M_N-1} |\hat{a}(jM_N + u, v)|^2 \le \beta^{2/p-2} m_N^{1-p/2} M_N^{2-2/p} \times$$

$$\times \sum_{\substack{l=0\\ M_N \le \beta M_l}}^{N-1} M_N^{2-2/p} M_N^{-2} \sum_{j=1}^{m_N-1} \sum_{u,v=0}^{M_N-1} |\hat{a}(jM_N + u, v)|^2 \le$$

$$\le \beta^{2/p-2}(N - l_*) m_N^{1-2/p} M_N^{2-4/p} \|a\|_2^2 \le \beta^{2/p-2}(N - l_*) \times$$

$$\times m_N^{1-2/p} M_N^{2-4/p} \gamma^{2-4/p} M_{N+1}^{4/p-2} \le \beta^{2/p-2}(N - l_*) m_N^{2/p-1},$$

where $l_* = 0, ..., N-1$ and $M_N \le \beta M_{l_*}$ but $M_N > \beta M_{l_*-1}$. (If $M_N > \beta M_l$ for all $l = 0, ..., N-1$ then $\sum^{(21)} = 0$.) The assumption $M_N \le \beta M_{l_*}$ implies $2^{N-l_*} \le m_{l_*} \cdot ... \cdot m_{N-1} \le \beta$. In other words $m_{N-1} \le \beta$ and $N - l_* \le \log_2 \beta$. Moreover, $m_N \le q\beta$ since m is power-like. Consequently, $\sum^{(21)} \le \beta^{2/p-2}(q\beta)^{2/p-1} \log_2 \beta$.

In order to estimate $\sum^{(22)}$ let $\nu \in \mathcal{U}_I$ and

$$a_\nu(\tau, t) := \begin{cases} a(\tau, t) & (\tau \in I_N(x_I, \nu), t \in J) \\ 0 & ((\tau, t) \in G^2 \setminus (I_N(x_I, \nu) \times J)). \end{cases}$$

Then $|\hat{a}(jM_k, nM_l)| = |\hat{a}(jM_k, 0)| = |\hat{a}(jM_k, v)|$ and $|\hat{a}_\nu(jM_k, v)| = |\hat{a}_\nu(jM_k + u, v)|$ if $u = 0, ..., M_{N+1} - 1, N < k \in \mathbb{N}, j = 1, ..., m_k - 1$ and $v = 0, ..., M_N - 1, N > l \in \mathbb{N}, n = 1, ..., m_l - 1$. Hence

$$\sum^{(22)} = \sum_{k=N+1}^{\infty} \sum_{\substack{l=0\\ M_k \le \beta M_l}}^{N-1} (m_k m_l)^{1-\frac{2}{p}} (M_k M_l)^{2-\frac{2}{p}} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\hat{a}(jM_k, nM_l)|^2 =$$

$$= \sum_{k=N+1}^{\infty} \sum_{\substack{l=0\\ M_k \le \beta M_l}}^{N-1} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} (m_l - 1) M_N^{-1} \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{v=0}^{M_N-1} |\hat{a}(jM_k, v)|^2 =$$

$$= M_N^{-1} \sum_{k=N+1}^{\infty} \sum_{\substack{l=0 \\ M_k \le \beta M_l}}^{N-1} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} (m_l - 1) \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{v=0}^{M_N-1} |\sum_{\nu \in \mathcal{U}_I} \hat{a}_\nu (jM_k, v)|^2 \le$$

$$\le \gamma M_N^{-1} \sum_{k=N+1}^{\infty} \sum_{\substack{l=0 \\ M_k \le \beta M_l}}^{N-1} (m_k m_l)^{1-2/p} m_l (M_k M_l)^{2-2/p} \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{v=0}^{M_N-1} \sum_{u=0}^{M_{N+1}-1} M_{N+1}^{-1} \sum_{\nu \in \mathcal{U}_I} |\hat{a}_\nu (jM_k + u, v)|^2 \le$$

$$\le \gamma M_N^{-1} M_{N+1}^{-1} M_{N+1}^{2-2/p} \sum_{k=N+1}^{\infty} \sum_{l=0, M_k \le \beta M_l}^{N-1} (M_k/\beta)^{2-2/p} \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{v=0}^{M_N-1} \sum_{u=0}^{M_{N+1}-1} \sum_{\nu \in \mathcal{U}_I} |\hat{a}_\nu (jM_k + u, v)|^2 \le$$

$$\le \gamma M_N^{-1} M_{N+1}^{-1} M_{N+1}^{2-2/p} (M_{N+1}/\beta)^{2-2/p} \sum_{k=N+1}^{\infty} (N - l^{(k)}) \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{v=0}^{M_N-1} \sum_{u=0}^{M_{N+1}-1} \sum_{\nu \in \mathcal{U}_I} |\hat{a}_\nu (jM_k + u, v)|^2,$$

where $l^{(k)} = 0, ..., N-1$ and $M_k \le \beta M_{l^{(k)}}$ but $M_k > \beta M_{l^{(k)}-1}$ $(k = = N+1, ...)$. Analoguosly to the previous cases we continue by noting that $2^{N-l^{(k)}+1} \le m_{l^{(k)}} \cdot ... \cdot m_N \le m_{l^{(k)}} \cdot ... \cdot m_{k-1} \le \beta$ $(k = N+1, ...)$, i.e. $m_N \le \beta$ and $N - l^{(k)} \le \log_2 \beta - 1 =: C_\beta$. Therefore it follows that

$$\sum\nolimits^{(22)} \le C_\beta \gamma \beta^{2/p-2} M_N^{-1} M_{N+1}^{3-4/p} \|a\|_2^2 \le$$

$$\le C_\beta \gamma^{3-4/p} M_N^{-1} M_{N+1}^{3-4/p} M_{N+1}^{4/p-2} \le C_\beta m_N \le \beta C_\beta .$$

Finally let $\sum^{(23)}$ be investigated. We recall the decomposition $a = = \sum_{\substack{\nu \in \mathcal{U}_I \\ \mu \in \mathcal{U}_J}} a_{\nu,\mu}$. Thus (see the analogous observations with respect to $\sum^{(12)}$)

$$\sum\nolimits^{(23)} = \sum_{k=N+1}^{\infty} \sum_{\substack{l=N+1\\ M_k\le\beta M_l}}^{k-1} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\hat{a}(jM_k, nM_l)|^2 =$$

$$= \sum_{k=N+1}^{\infty} \sum_{\substack{l=N+1\\ M_k\le\beta M_l}}^{k-1} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\sum_{\nu\in\mathcal{U}_I} \sum_{\mu\in\mathcal{U}_J} \hat{a}_{\nu,\mu}(jM_k, nM_l)|^2 \le$$

$$\le \gamma^2 M_{N+1}^{-2} \sum_{k=N+1}^{\infty} \sum_{\substack{l=N+1\\ M_k\le\beta M_l}}^{k-1} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} \times$$

$$\times \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} \sum_{\nu\in\mathcal{U}_I} \sum_{\mu\in\mathcal{U}_J} \sum_{u,v=0}^{M_{N+1}-1} |\hat{a}_{\nu,\mu}(jM_k+u, nM_l+v)|^2 \le$$

$$\le \gamma^2 M_{N+1}^{-2} M_{N+1}^{4-4/p} \sum_{k=N+1}^{\infty} \sum_{\substack{l=N+1\\ M_k\le\beta M_l}}^{k-1} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} \sum_{\nu\in\mathcal{U}_I} \sum_{\mu\in\mathcal{U}_J} \times$$

$$\times \gamma^2 M_{N+1}^{2-4/p} \|a\|_2^2 \le \gamma^{4-4/p} \le 1.$$

The sum $\sum^{(3)}$ can be estimated in a similar way as $\sum^{(2)}$ which completes the proof of Th. 2. $\Diamond$

It is not hard to see that Th. 2 fails to hold if the sum $\sum_\alpha^\beta$ is replaced by $\sum_{k,l=0}^\infty$. In other words if $0 < p \le 1$ then the inequality

$$\Big(\sum_{k,l=0}^{\infty} (m_k m_l)^{1-2/p} (M_k M_l)^{2-2/p} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\hat{f}(jM_k, nM_l)|^2 \Big)^{1/2} \le C_p \|f\|_{H^p}$$

cannot be true for all $f \in H^p(G)$. Indeed, let $m_n := 2 \quad (n \in \mathbb{N})$ (i.e. consider the double Walsh system) and for all $N \in \mathbb{N}$ define a_N as

$$a_N(x,y) := 2^{2N(1/p-1)} D_{2^N}(x) r_N(y) D_{2^N}(y) \qquad \big((x,y) \in G^2\big).$$

Then (see (2)) a_N is a p-atom and

$$\sum_{k,l=0}^{\infty} 2^{2(k+l)(1-1/p)}|\hat{a}_N(2^k,2^l)|^2 = 2^{4N(1/p-1)} \sum_{k=0}^{N-1} 2^{2(k+N)(1-1/p)} \geq N$$

if $p = 1$ or $2^{2N(1/p-1)}$ if $0 < p < 1$, i.e. $\sup_N \sum_{k,l=0}^{\infty} 2^{2(k+l)(1-1/p)} \times$ $\times|\hat{a}_N(2^k,2^l)|^2 = \infty$.

Now we show that the condition that m is power-like plays an important role in Th. 2. Let $N \in \mathbb{N}$ and consider the function

$$f_N(x,y) := \big(D_{M_{N+1}}(x) - D_{M_{N+1}}(\tilde{x})\big)\big(D_{M_{N+1}}(y) + D_{M_{N+1}}(\tilde{y})\big) \qquad \big((x,y) \in G^2\big),$$

where for each $z = (z_0, z_1, \ldots) \in G_m$ we define the element $\tilde{z} \in G_m$ as $\tilde{z} := (z_0, \ldots, z_{N-1}, z_N - 1 \,(\mathrm{mod}\, m_N), z_{N+1}, \ldots)$. Taking into account (2) it follows that the support of f is $I := \big(I_{N+1}(0) \cup I_N(0,1)\big) \times \big(I_{N+1}(0) \cup$ $\cup I_N(0,1)\big)$. By a suitable choice of m it can be assumed that I is an interval. Furthermore, a simple calculation shows that

$$f_N(x,y) = \Big(\sum_{j=1}^{m_N-1}(1-\exp\frac{2\pi ij}{m_N})\sum_{k=jM_N}^{(j+1)M_N-1}\Psi_k(x)\Big)\times$$

$$\times\Big(2D_{M_N}(y) + \sum_{n=1}^{m_N-1}(1+\exp\frac{2\pi in}{m_N})\sum_{l=nM_N}^{(n+1)M_N-1}\Psi_l(y)\Big) \qquad \big((x,y) \in G^2\big).$$

Let $0 < p < 1$ and $a_N := 2^{-2/p}M_{N+1}^{2/p-2}f_N$. Then a_N is a p-atom and $|\hat{a}_N(jM_N, nM_l)| = 2^{1-2/p}M_{N+1}^{2/p-2}|1-\exp\frac{2\pi ij}{m_N}|$ if $j = 1, \ldots, m_N-1; l =$ $= 0, \ldots, N-1$ and $n = 1, \ldots, m_l - 1$. Therefore if $M_N \leq \beta M_{N-1}$, i.e. $m_{N-1} \leq \beta$ then (see the proof of Th. 2)

$$\sum{}^{(21)} = 2^{2-4/p}M_{N+1}^{4/p-4} \times$$

$$\times \sum_{\substack{l=0\\ M_N\leq\beta M_l}}^{N-1}(m_Nm_l)^{1-2/p}(M_NM_l)^{2-2/p}\sum_{j=1}^{m_N-1}\sum_{n=1}^{m_l-1}|1-\exp\frac{2\pi ij}{m_N}|^2 \geq$$

$$\geq C_pm_N^{2/p-3}M_N^{2-2/p}\sum_{\substack{l=0\\ M_N\leq\beta M_l}}^{N-1}M_{l+1}^{2-2/p}\sum_{1\leq j\leq m_N/2}j^2/m_N^2 \geq C_pm_N^{2/p-2}.$$

The last inequality shows that $\sum^{(21)}$ can be not bounded if the sequence m is not power-like.

On the other hand if we replace the Hardy space $H^p(G)$ $(0 < p \leq 1)$ by $H^p_\sigma(G)$ in Th. 2 then the assumption on m can be omitted. Namely, the following theorem is true.

Theorem 3. *Let $0 < p \leq 1, 0 < \alpha \leq 1 \leq \beta$ are given. Then there exists a constant C depending only on p, m, α, β such that*

$$\Big(\sum\nolimits_\alpha^\beta (m_k m_l)^{1-2/p}(M_k M_l)^{2-2/p} \sum_{j=1}^{m_k-1}\sum_{n=1}^{m_l-1} |\hat{f}(jM_k, nM_l)|^2\Big)^{1/2} \leq C\|f\|_{H^p_\sigma}$$

holds for all $f \in H^p_\sigma(G)$.

By Th. 1 it is enough to show (3) where the supremum is taken now over all simple p-atoms a. Formally we can write m_N instead of γ in the proof of Th. 2, that is, we have $\|a\|_2 \leq M_N^{2/p-1}$. This means that the assumption $m_{n+1} \leq q m_n$ $(n \in \mathbb{N})$ is not needed.

Similarly, if we consider the special case $\alpha = \beta = 1$ then we get

Theorem 4. *If $0 < p \leq 1$ then there exists a constant C depending only on p, m such that*

$$\Big(\sum_{k=0}^{\infty} m_k^{2-4/p} M_k^{4-4/p} \sum_{j=1}^{m_k-1} |\hat{f}(jM_k, nM_k)|^2\Big)^{1/2} \leq C\|f\|_{H^p} \quad (f \in H^p(G)).$$

Indeed, in the proof of the estimation $\sum^{(1)} \leq C_p$ (see the proof of Th. 2) we have not used the power-like condition of m.

Finally, we formulate the dual version of Th. 2.

Theorem 5. *Assume that m is power-like and $0 < \alpha \leq 1 \leq \beta$. Furthermore, let $\alpha_{k,l}$ $(k, l \in \mathbb{N})$ be real or complex numbers such that*

$$\sum\nolimits_\alpha^\beta m_k m_l \sum_{j=1}^{m_k-1}\sum_{n=1}^{m_l-1} |\alpha_{jM_k,nM_l}|^2 < \infty.$$

Then the function $f := \sum_\alpha^\beta m_k m_l \sum_{j=1}^{m_k-1}\sum_{n=1}^{m_l-1} \alpha_{jM_k,nM_l}\Psi_{jM_k,nM_l}$ belongs to $BMO(G)$ and

$$\|f\|_{BMO} \leq C\Big(\sum\nolimits_\alpha^\beta m_k m_l \sum_{j=1}^{m_k-1}\sum_{n=1}^{m_l-1} |\alpha_{jM_k,nM_l}|^2\Big)^{1/2},$$

where the constant C depends only on m, α, β.

Taking into consideration Th. 3 we get the $\mathcal{BMO}$-variant of Th. 5, i.e., Th. 5 will be true for all m if we replace BMO by $\mathcal{BMO}$.

In the bounded case, i.e. when $\sup_n m_n < \infty$ the factors m_k, m_l in Th. 5 can obviously be omitted. Since $\|.\|_2 \leq \|.\|_{BMO}$, a Kintchin type inequality follows:

Corollary 1. *Suppose that m is bounded, $0 < \alpha \le 1 \le \beta$ and f is of the form*

$$f = \sum\nolimits_{\alpha}^{\beta} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} \hat{f}(jM_k, nM_l)\Psi_{jM_k,nM_l}.$$

Then $C\|f\|_{BMO} \le \left(\sum_{\alpha}^{\beta} \sum_{j=1}^{m_k-1} \sum_{n=1}^{m_l-1} |\hat{f}(jM_k, nM_l)|^2\right)^{1/2} \le \|f\|_{BMO}$ *with a constant $C > 0$ independent on f.*

References

[1] COIFMAN, R.F. and WEISS, G.: Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[2] PALEY, R.E.A.C.: A remarkable series of orthogonal functions, *Proc. London Math. Soc.* **34** (1932), 241–279.

[3] SCHIPP, F., WADE, W.R., SIMON, P. and PÁL, J.: Walsh series: An introduction to dyadic harmonic analysis, Adam Hilger, Bristol-New York, 1990.

[4] SIMON, P.: Investigations with respect to the Vilenkin system, *Annales Univ. Sci. Budapest Eötvös, Sect. Math.* **28** (1985), 87–101.

[5] SIMON, P. and WEISZ, F.: Paley type inequalities for Vilenkin-Fourier coefficients, *Acta Sci. Math. (Szeged)* **63** (1997), 107–124.

[6] SIMON, P. WEISZ, F.: Hardy type inequalities for two-parameter Vilenkin-Fourier coefficients, *Studia Math.* **125**/3 (1997), 231–246.

[7] VILENKIN, N.Ja.: On a class of complete orthonormal systems, *Izv. Akad. Nauk SSSR, Ser. Math.* **11** (1947), 363–400 (Russian), English transl.: *Amer. Math. Soc. Transl.* **28** (1963), 1–35.

[8] WEISZ, F.: Martingale Hardy spaces and their applications in Fourier-analysis, Lectures Notes in Math. **1568**, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[9] WEISZ, F.: An application of two-parameter martingales in harmonic analysis, *Studia Math.* **107** (1993), 115–126.