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ON ASYMPTOTICALLY CORRELATED q-MULTIPLICATIVE FUNCTIONS

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Abstract: Let $k \geq 1$, $a_1, \ldots, a_k \in \mathbb{N}$, $b_1, \ldots, b_k \in \mathbb{N}_0$, $q \geq 2$, $f_1, \ldots, f_k$ be such complex valued q-multiplicative functions for which

$$L(n) := \alpha_1 f_1(a_1 n + b_1) + \ldots + \alpha_k f_k(a_k n + b_k)$$

tends to zero for almost all n, with a suitable nontrivial choice of complex coefficients $(\alpha_1, \ldots, \alpha_k)$. Assume that no proper subsystem of $f_1, \ldots, f_k$ satisfies this condition. The following assertions are proved: If $k = 1$, then either $f(an+$ $+b) = 0$ for all $n \in \mathbb{N}$, or $f(n) \to 0$ for almost all $n \in \mathbb{N}$. If $k \geq 2$, then $L(n) = 0$ identically, and there is an integer $R \geq 0$ such that

$$f_1(a_1 m q^R) = \ldots = f_k(a_k m q^R) \neq 0$$

holds for every $m \in \mathbb{N}_0$. If there exist i and j, $i \neq j$ such that $(a_i, q) = (a_j, q) =$ $= 1$, then

$$f_l(mq^R) = z_l^m \ (m \in \mathbb{N}_0), \ l = 1, \ldots, k,$$

where $z_1, \ldots, z_k \in \mathbb{C}$, such that $z_1^{a_1} = \ldots = z_k^{a_k} \ (\neq 0)$.

1. Introduction

Let $q \geq 2$ be an integer and $A_q = \{0, 1, ..., q-1\}$. We shall use the standard notations: $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the set of positive integers, nonnegative integers, integers, real-numbers, complex numbers, respectively. The *q-ary expansion* of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_j(n) \in A_q$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j \tag{1.1}$$

holds. $\varepsilon_j(n)$ are called the *digits in the q-ary expansion of n.*

Let $\mathcal{M}_q$ be the set of complex-valued q-multiplicative, and $\mathcal{A}_q$ be the set of real-valued q-additive functions. A function $f : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathcal{M}_q$, if $f(0) = 1$ and for every $n \in \mathbb{N}_0$,

$$f(n) = \prod_{j=0}^{\infty} f(\varepsilon_j(n) q^j). \tag{1.2}$$

A function $g : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$, if $g(0) = 0$, and for every $n \in \mathbb{N}_0$,

$$g(n) = \sum_{j=0}^{\infty} g(\varepsilon_j(n) q^j). \tag{1.3}$$

Since $f(\varepsilon_j(n)q^j) = 1$, $g(\varepsilon_j(n)q^j) = 0$ for all those j for which $q^j > n$, therefore the summands on the right hand side of (1.3), and the factors on the right hand side of (1.2) are finite.

The following remarks are obvious.

(1) If $g \in \mathcal{A}_q$, $z \in \mathbb{C}$, then $f(n) := z^{g(n)} \in \mathcal{M}_q$.

(2) If $f \in \mathcal{M}_q$ and f takes on only positive real values, then $g(n) := \log f(n)$ belongs to $\mathcal{A}_q$.

(3) The linear function $g(n) = cn$ belongs to $\mathcal{A}_q$, $f(n) := z^n$ belongs to $\mathcal{M}_q$, for every $q \geq 2$.

(4) If $f \in \mathcal{M}_q$, then $f \in \mathcal{M}_{q^k}$, and if $g \in \mathcal{A}_q$, then $g \in \mathcal{A}_{q^k}$, $k = 1, 2, ...$

(5) Let $f_j(n) := f(nq^j)$, $g_j(n) := g(nq^j)$, $j = 1, 2, ...$ If $f \in \mathcal{M}_q$, then $f_j \in \mathcal{M}_q$, if $g \in \mathcal{A}_q$, then $g_j \in \mathcal{A}_q$.

(6) $\mathcal{A}_q$ is a linear space.

The notion of the q-additive function can be extended to an arbitrary Abelian group G. Then $\mathcal{A}_q(G)$ (the class of G-valued q-additive functions) consists of those $g : \mathbb{N}_0 \to \mathbb{G}$ for which $g(0) = 0$ and (1.3) holds. It is obvious furthermore that $g(n) := n\alpha$ belongs to $\mathcal{A}_q(G)$ for every choice of $\alpha \in \mathbb{G}$.

A sequence $\{x_n\}$ $n \in \mathbb{N}_0$ of real or complex numbers is said to converge in frequency (or, for almost all $n \in \mathbb{N}_0$) to some y, if

$$\frac{1}{N}\#\{n < N \| x_n - y| > \delta\} \to 0 \quad (N \to \infty)$$

for every $\delta > 0$. Similarly, if G is a topological Abelian group, and x_n is an infinite sequence in G, then we say that x_n converges to some $y \in G$, if for every open set U containing 0,

$$\frac{1}{N}\#\{n < N | x_n - y \notin U\} \to 0 \quad (N \to \infty).$$

The notion of q-additive functions was introduced by A.O. Gelfond [1]. H. Delange [2] gave necessary and sufficient conditions in order that some $u \in \mathcal{A}_q$ would have a limit distribution. Kátai [3] proved that the same conditions are both necessary and sufficient if we consider the frequencies of the values of $u \in \mathcal{A}_q$ on the set of primes. There are a lot of interesting open problems with respect to the value distribution of q-additive functions. One of the simplest is the following:

Let $q = 2$, $f \in \mathcal{A}_2$, and assume that

$$\limsup_{x\to\infty} \frac{1}{x}\#\{\, n \le x \mid |f(3n) - f(n)| > K \,\} = C(K),$$

$C(K) \to 0$ as $K \to \infty$. Is it true that

$$\sum_{j=1}^{\infty} f^2(2^j) < \infty?$$

Our papaer is the first attempt to solve such kind of problems.

2. Formulation of the problem

Let $f_j \in \mathcal{M}_q$, $a_j \in \mathbb{N}$, $b_j \in \mathbb{N}_0$, $j = 1, \ldots, k$. We say that the functions $\{f_j(a_j n + b_j)\}$ $j = 1, \ldots, k$ are *asymptotically correlated* if there exist non-identically zero complex numbers $\alpha_1, \ldots, \alpha_k$ for which

$$L(n) := \sum_{j=1}^{k} \alpha_j f_j(a_j n + b_j) \tag{2.1}$$

tends to zero for almost all integer as $n \to \infty$.

We say that the correlation is *non-reducible* if no proper subset of $\{f_j(a_j n + b_j)\}$ is asymptotically correlated.

The following assertions are clear.

(1) Assume that the functions $\{f_j(a_j n + b_j)\}(j = 1, \ldots, k)$ are correlated, and that for some $l \in \{1, \ldots, k\}$, $f_l(a_l n + b_l) \to 0$ $(n \to \infty)$ for almost all n. Then the correlation holds, if we drop f_l.

(2) Keeping the notations, let M be the set of those vectorials $(\alpha_1, \ldots, \alpha_k)$ over $\mathbb{C}$ for which $L(n)$ defined by (2.1) tends to zero for almost

all n. Then M is a subspace in $\mathbb{C}^k$. $M = \{0\}$, if $\{f_j\}$ are not correlated, $\dim M = 1$ if and only if they are non-reducibly correlated.

Let $a_1, a_2 \in \mathbb{N}$, $f_1, f_2 \in \mathcal{M}_q$. We say that the functions $f_1(a_1 n)$, $f_2(a_2 n)$ *belong to the same class* if there is a suitable integer R for which $f_1(a_1 m q^R) = f_2(a_2 m q^R)$ holds for every $m \in \mathbb{N}_0$.

We shall prove the following theorems.

Theorem 1. *Let $f \in \mathcal{M}_q$, $a > 0$, $b \geq 0$ such that $f(an+b) \to 0$ for almost all $n \in \mathbb{N}$. Assume that there exists an $n_0 \in \mathbb{N}_0$, for which $f(an_0 + b) \neq 0$. Then $f(n) \to 0$ as $n \to \infty$, for almost all n.*

The proof is an easy consequence of the following

Lemma 1. *Let $f \in \mathcal{M}_q$, $\mathcal{B} = \{cq^j | f(cq^j) = 0,\ c \in A_q,\ j = 0, 1, \ldots\}$. If $\mathcal{B}$ is an infinite set, then $f(n) = 0$ for almost all $n \in \mathbb{N}_0$.*

Theorem 2. *Let $k \geq 2$, and assume that the functions $f_j \in \mathcal{M}_q$ and $f_j(a_j n + b_j)$ $j = 1, \ldots, k$ are asymptotically correlated,* (2.1) *holds and that the correlation is non-reducible. Then the functions $f_j(a_j n + b_j)$ belong to the same class, and $f_j(a_j m q^R) \neq 0$ for every $m \in \mathbb{N}_0$, if R is sufficiently large. Furthermore $L(n) = 0$ holds identically, i.e. for every $n \in \mathbb{N}_0$.*

Assume additionally that at least two of $a_1, \ldots, a_k$ are coprime to q. Then there exist complex numbers $z_1, \ldots, z_k$ such that $|z_1| \geq 1$, $z_1^{a_1} = z_2^{a_2} = \ldots = z_k^{a_k}$, and that for a suitable large integer S, $f_j(mq^S) = z_j^m$ $(m \in \mathbb{N}_0)$, $j = 1, \ldots, k$.

To ease the proof of the first assertion we shall prove Lemma 2., the last assertion will follow from Ths. 3, 4.

Lemma 2. *Let $\beta_1, \ldots, \beta_h$ be nonzero complex numbers, $a_j \in \mathbb{N}$, $g_j \in \mathcal{M}_q$, $j = 1, \ldots, h$, $S(n) = \beta_1 g_1(a_1 n) + \ldots + \beta_h g_h(a_h n)$. Assume that $g_j(n) \neq 0$ $(n \in \mathbb{N}_0,\ j = 1, \ldots, k)$, and that no two of $\{g_j(a_j n)\}$ do belong to the same class. If $S(n) \to 0$ for almost all $n \in \mathbb{N}_0$, then $g_j(n) \to 0$ for almost all n and for every $j = 1, \ldots, k$.*

Theorem 3. *Let G be an Abelian group, $a, b \in \mathbb{N}$ such that $(ab, q) = 1$, $a \neq b$. Let $u, v \in \mathcal{A}_q(G)$ be such that*

$$u(a\ n) = v(b\ n) \quad (n \in \mathbb{N}_0).$$

Then there exists a suitable $R \in \mathbb{N}$, $\alpha, \beta \in G$, such that $u(q^R m) = m\alpha$, $v(q^R m) = m\beta$, furthermore $a\alpha = b\beta$.

Theorem 4. *Let G be an Abelian group, $a \in \mathbb{N}$, $(a, q) = 1$, $u, v \in \mathcal{A}_q(G)$ such that*

$$u(an) = v(an) \quad (n \in \mathbb{N}_0).$$

Then $\delta(n) := v(n) - u(n)$ satisfies $\delta(n) = n\beta$, where $\beta \in G$, $a\beta = 0$. The converse assertion is true as well.

3. Proof of Lemma 1 and Theorem 1

If $cq^j \in \mathcal{B}$, and $f(n) \neq 0$, then $\varepsilon_j(n) \neq cq^j$. Thus

$$\#\{n < q^N,\ f(n) \neq 0\} = \prod_{j=0}^{N-1} \{ \sum_{cq^j \notin \mathcal{B}} 1 \}$$

and the right hand side is $o(q^N)$ as $N \to \infty$ if $\mathcal{B}$ is infinite.

To prove Th. 1, we may assume that $\mathcal{B}$ is finite. Let s be so large that $cq^j \notin \mathcal{B}$, if $j \geq s$. We shall write a as $a_1\xi$, where $(a_1, q) = 1$, and the prime divisors of ξ divide q. Let T be so large that $\xi | q^T$. Since $f(a(n_0 + q^R m) + b) \to 0$ for almost all m, and

$$f(a(n_0 + q^R m) + b) = f(an_0 + b) f(aq^R m), \quad \text{if} \quad an_0 + b < q^R,$$

furthermore $\xi | q^T$, therefore $f(a_1 q^{T+R} n) \to 0$ for almost all n. Let $H = \max(s, T + R)$, $\varphi(n) := f(q^{T+R} n)$. Then $\varphi(a_1 n) \to 0$ for almost all n. Let $a_1 < q^M$. If $n = u + q^M v$, $0 \leq u < q^M$, then choose $\widetilde{u} \in [0, a_1 - 1]$ be so that $\widetilde{u} + q^M v \equiv 0 \pmod{a_1}$, and let $S(n) := \widetilde{u} + q^M v$. It is clear that $|\varphi(n)| \leq c|\varphi(S(n))|$ and every fixed value $S(n)$ occurs at most for q^R integers. Hence we obtain that $\varphi(n) \to 0$ for almost all n, and this implies the assertion of the theorem readily. ◊

4. Proof of Lemma 2

The assertion is true for $h = 1$, see Th. 1. We shall use induction on h. Assume it is true if the number of functions is less than h. Let $\overline{\xi}(n) = (\beta_1 g_1(a_1 n), ..., \beta_h g_h(a_h n))$ $(n \in \mathbb{N}_0)$. If the vectorials $\overline{\xi}(n)$ belong to a one-dimensional subspace then they are parallel to $\overline{\xi}(0)$, thus $g_1(a_1 n) = ... = g_h(a_h n)$ $(n \in \mathbb{N}_0)$, and so the functions belong to the same class. Assume that there is an n_0 for which $\overline{\xi}(n_0)$ is not parallel to $\overline{\xi}(0)$. Let R be so large that $(\max_h a_j)\, n_0 < q^R$. Then

$$S(n_0 + mq^R) = \sum_{j=1}^{h} \beta_j g_j(a_j n_0) g_j(a_j m q^R), \quad S(mq^R) = \sum_{j=1}^{h} \beta_j g_j(a_j m q^R),$$

and so

$$S(n_0 + mq^R) - g_l(a_l n_0) S(mq^R) = \sum_{j=1}^{h} \beta_j (g_j(a_j n_0) - g_l(a_l n_0)) g_j(a_j m q^R),$$

and this sequence tends to zero for almost all $m \to \infty$. The right hand side is non-empty. The number of the functions with nonzero coefficients is less than h. Thus $g_j(a_j m q^R) \to 0$ for almost all m, whenever j is such an index for which $g_j(a_j n_0) \neq g_l(a_l n_0)$. Since l was arbitrary, there exists such a j. Then $g_j(n) \to 0$ for almost all n, and we reduced the number of the functions. The proof is complete. ◊

5. Proof of Theorem 2

Assume that the conditions hold. Let $I_1, I_2, \dots, I_s$ be the partition of the set $\{1, 2, \dots, k\}$ according to the classification of the functions $f_j(a_j n)$. Let R be so large that $f_i(a_i m q^R) = f_j(a_j m q^R)$ for every $m \in \mathbb{N}_0$ and for every such pair f_i, f_j which belong to the same class, furthermore $f_i(a_i m q^R) \neq 0$ ($m \in \mathbb{N}_0$, $i = 1, \dots, k$).

Let $L_h(n) := \sum_{i \in I_h} \alpha_i f_i(a_i n + b_i)$. Then $L(n) = \sum_{h=1}^{s} L_h(n)$.

Assume that the indices of the functions are so chosen that $j \in I_j$ ($j = 1, \dots, s$). Let $T \geq R$, $\max(a_i n + b_i) < q^T$. For such T we have that

$$L_h(n + mq^T) = L_h(n) f_h(a_h q^T m).$$

We shall deduce that $L_h(n) = 0$ for every $h = 1, \dots, s$, and since n was arbitrary, it holds identically. Indeed, assume indirectly that $L_h(n) \neq 0$ if $h = j_1, j_2, \dots, j_t$. Then

$$\sum_{l=1}^{t} L_{j_l}(n) f_{j_l}(a_{j_l} m q^T) \to 0 \quad \text{for almost all } m.$$

We can apply Lemma 2., which implies that $f_{j_l}(a_{j_l} m q^T) \to 0$ for almost all m, which by Th. 1 implies that $f_j(n) \to 0$ for almost all n. This contradicts to the assumption that the correlation is non-reducible. Furthermore we obtain that $s = 1$.

To prove the last assertion we start from the relation $f_1(a_1 m q^R) = = \dots = f_k(a_k m q^R)$ $m \in \mathbb{N}_0$), and from the assumption $f_j(a_j m q^R) \neq 0$. Then, let $\gamma_j(n) = \log |f_j(nq^R)|$, $\arg f_j(nq^R) = 2\pi \kappa_j(n)$. We have that $\gamma_1(a_1 n) = \gamma_2(a_2 n) = \dots = \gamma_k(a_k n)$ and

$$\kappa_1(a_1 n) \pmod 1 = \kappa_2(a_2 n) \pmod 1 = \dots = \kappa_k(a_k n) \pmod 1.$$

Th. 3 for $G = \mathbb{R}$, and for $G = T$ = one-dimensional torus implies the last assertion. $\Diamond$

6. Proof of Theorem 3

Assume that $a < b$. By changing q to q^k if necessary, we may assume that $ab < q$. Let $Q = q^m$, where m is a suitable positive integer which will be specified later.

Let $0 \leq m_0 < a$, l_0 be the least positive integer for which $Qm_0 + l_0 \equiv \equiv 0 \pmod a$. Let $Qm_0 + l_0 = as$. Then $0 \leq s < Q$. Let $bs = hQ + r$, $0 < < r < Q$. We shall define the integers u_j, k_j ($j = 0, \dots, j^*$) from the relation $r + bj = k_j Q + u_j$, $0 \leq u_j < Q$, where $j^* = [\frac{Q}{a}] - 1$. Observe that $l_0 + ja < Q$ if $0 \leq j \leq j^*$. Consequently for every $t \in \mathbb{N}_0$,

$$u(Q(m_0+at))+u(l_0+ja)=u((Qm_0+l_0)+a(j+Qt))=$$
$$=u(as+a(j+Qt))=v(bs+bQt+bj)=$$
$$=v(hQ+r+bj+bQt)=v((h+bt+k_j)Q+u_j).$$

Assume that $j<j^*$, and apply this for $j+1$ instead of j, as well. Then,

$$u(l_0+(j+1)a)-u(l_0+ja)= \tag{6.1}$$
$$=v((h+bt+k_j)Q+(k_{j+1}-k_j)Q+u_{j+1})-v((h+bt+k_j)Q+u_j).$$

In the sequence $k_0,k_1,...,k_{j^*}$, the difference $k_{\nu+1}-k_\nu$ is zero or one. $k_{j^*}\sim\frac{b}{a}\frac{Q}{a}>1$ if M is large enough. Let j_1 be that value for which $k_{j_1}=0$, $k_{j_1+1}=1$. Let $\delta(l_0,j_1):=u(l_0+(j_1+1)a)-u(l_0+j_1a)$. From (6.1) we obtain that

$$\delta(l_0,j_1)=v((h+bt+1)Q+u_{j_1+1})-v((h+bt)Q+u_{j_1}). \tag{6.2}$$

The left hand side does not depend on t. Let $N>M$. For every $P\in$ $\in[0,q^N-2]$ there is an integer t for which $h+bt\equiv P \pmod{q^N}$. Let $h+bt=P+q^N\lambda$. Then

$$v((h+bt+1)Q+u_{j_1+1})=v(q^NQ\lambda)+v((P+1)Q)+v(u_{j_1+1}),$$
$$v((h+bt)Q+u_{j_1})=v(q^NQ\lambda)+v(PQ)+v(u_{j_1}),$$
$$\delta(l_0,j)=v((P+1)Q)-v(PQ)+(v(u_{j_1+1})-v(u_{j_1})).$$

Thus $v((P+1)Q)-v(PQ)=c=v(1\cdot Q)$, consequently $v(PQ)=Pv(Q)$.

Let $\varphi(n)=u(nQ)$, $\psi(n)=v(nQ)$. Then $\psi(1)=v(Q)$, $\psi(n)=$ $=n\psi(1)$. Since $\varphi(an)=\psi(bn)$, therefore $\varphi(an)=bn\psi(1)$.

Let $m\in\mathbb{N}_0$, $N\in\mathbb{N}$, $0\le l_0<a$ be such that $q^Nm+l_0\equiv$ $\equiv 0 \pmod a$, $l_j=l_0+ja$. Then for $l_j<q^N$ we have $\varphi(q^Nm)=$ $=\varphi(q^Nm+l_j)-\varphi(l_j)=\psi(\frac{q^Nm+l_j}{a}b)-\varphi(l_j)=\frac{q^Nm+l_j}{a}b\psi(1)-\varphi(l_j)=$ $=\frac{q^Nm+l_0}{a}+\{jb\psi(1)-\varphi(l_0+ja)\}$. Since the left hand side does not depend on l_j, therefore

$$\varphi(l_0+ja)=\varphi(l_0)+jb\psi(1)=\varphi(l_0)+\varphi(ja), \tag{6.3}$$

and this holds for every $j\in\mathbb{N}_0$, since N can be arbitrary large.

Let $n=t_1+qn_1$, $qn_1=t_2+Sa$, where $t_1+t_2<a$, $t_1,t_2>0$. From (6.3) we have that

$$\varphi(n)=\varphi(t_1+t_2+Sa)=\varphi(t_1+t_2)+\varphi(Sa),$$

and from the q-additivity, that

$$\varphi(n)=\varphi(t_1)+\varphi(qn_1)=\varphi(t_1)+\varphi(t_2)+\varphi(Sa).$$

Hence we obtain that $\varphi(t_1+t_2)=\varphi(t_1)+\varphi(t_2)$, i.e. that $\varphi(l)=l\varphi(1)$ $(l=1,\dots,a-1)$. If we choose $t_2=a-t_1$, we similarly get

$$\varphi(n) = \varphi((S+1)a), \quad \varphi(n) = \varphi(t_1) + \varphi(a - t_1) + \varphi(Sa),$$

whence

$$\varphi((S+1)a) - \varphi(Sa) = \varphi(t_1) + \varphi(a - t_1) = a\varphi(1),$$

and the left hand side equals

$$\psi((S+1)b) - \psi(Sb) = b\psi(1).$$

Thus $\varphi(l_0 + ja) = l_0\varphi(1) + ja\varphi(1)$ holds for every nonnegative integer $l_0 + ja$. We proved the theorem. ◊

7. Proof of Theorem 4

The last assertion is obvious, we shall prove the first one. By changing q to q^k if necessary we may assume that $a < q$. We start from $\delta(an) = 0$ $(n \in \mathbb{N}_0)$. Since for every integer of form qN_1 there is some $l \in [0, a-1]$ such that

$$qN_1 + l \equiv 0 \pmod{a},$$

therefore $\delta(qN_1) = \delta(qN_1 + l) - \delta(l) = -\delta(l)$. Thus the value $\delta(qN_1)$ depends only on $qN_1 \pmod{a}$. Let $d_0, d_1 \in A_q$ be so chosen that $qd_0 \equiv$ $\equiv t_0 \pmod{a}$, $q^2 d_1 \equiv t_1 \pmod{a}$, $t_0 + t_1 < a$, $t_0, t_1 > 0$. Thus

$$-\delta(t_1) - \delta(t_2) = \delta(qd_0) + \delta(q^2 d_1) = \delta(qd_0 + q^2 d_1) = -\delta(t_1 + t_2),$$

whence

$$\delta(t) = t\delta(1) \ (t = 0, 1, ..., a-1).$$

Similarly, if $t_1 + t_2 = a$, then

$$0 = \delta(t_1) + \delta(a - t_1) = (t_1 + (a - t_1))\delta(1) = a\delta(1).$$

This completes the proof. ◊

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