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INTEGRABILITY BEHAVIOUR OF THE MAXIMAL PARTIAL SUM OF ORTHOGONAL SERIES

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Abstract: The Rademacher–Menshov theorem is well known in the theory of orthogonal series. That is, if a sequence $\{a_k\}$ of real numbers satisfies $\sum_{k=1}^{\infty} a_k^2 \log^2 k < \infty$ then the orthogonal series $\sum_{k=1}^{\infty} a_k \phi_k(x)$ converges a.e. Moreover, the maximal partial sum $S_*(x) := \sup_{n\geq 1} |\sum_{k=1}^{n} a_k \phi_k(x)|$ belongs to L^2. Our main result states that in the case $\sum_{k=1}^{\infty} a_k^2 \log^2 k = \infty$ the maximal sum S_* does not belong to L^2 in general.

1. Introduction

Let $\{\phi_k(x) : k = 1, 2, \dots\}$ be a real-valued, orthonormal system (in abbreviation: ONS) on a positive measure space (X, μ). The celebrated Rademacher-Menshov theorem (see, e.g. [8, Vol. 2, p. 193]) states that if a sequence $\{a_k\}$ of real numbers is such that

$$\sum_{k=1}^{\infty} a_k^2 \log^2 k < \infty, \tag{1.1}$$

then the orthogonal series

$$\sum_{k=1}^{\infty} a_k \phi_k(x) \tag{1.2}$$

converges μ-almost everywhere (in abbreviation: a.e.), and the *maximal partial sum* (called the *majorant of the partial sums* in the Russian literature)

$$S_*(x) := \sup_{n\ge 1}\Big|\sum_{k=1}^{n} a_k\phi_k(x)\Big|$$

belongs to $L^2(X,\mu)$. More precisely, there exists an absolute constant C such that for all ONS $\{\phi_k(x)\}$ and for all sequences $\{a_k\}$ satisfying (1.1), we have

$$\|S_*\|_2 := \Big\{\int_X S_*^2(x)d\mu(x)\Big\}^{1/2} \le C\Big\{\sum_{k=1}^{\infty} a_k^2 \log^2(k+1)\Big\}^{1/2}. \tag{1.3}$$

In the sequel, by $C, C_1, C_2, \dots$ we denote positive constants. As usual, the logarithms are to the base 2, but any other base greater than 1 would do the same job.

The above convergence statement is exact. The second named author [4] proved that if the sequence $\{a_k\}$ of real numbers is such that $|a_1| \ge |a_2| \ge \dots$ and condition (1.1) is not satisfied, then there exists an ONS $\{\phi_k(x)\}$ on the unit interval $X := (0,1)$ (endowed with the Lebesgue measure) such that the orthogonal series (1.2) diverges a.e. The ONS $\{\phi_k(x)\}$ in the counterexample can be chosen to be uniformly bounded. Later on, Kashin [1] proved that this ONS $\{\phi_k(x)\}$ can be even chosen in such a way that each $\phi_k(x)$ takes on only the values $+1$ and -1 on the unit interval:

$$|\phi_k(x)| = 1 \qquad (k = 1, 2, \dots; x \in (0,1)). \tag{1.4}$$

A simplified proof of this last statement was provided by the second named author [7]. An ONS $\{\phi_k(x)\}$ on the interval (0,1) is called *sign type* if condition (1.4) is satisfied.

2. Sign type ONS

We shall prove that if condition (1.1) is not satisfied, then the maximal partial sum $S_*(x)$ does not belong to $L^2(X,\mu)$ in general. Even this is the case if we consider integrability L^p for some $p > 0$ instead of integrability L^2. This is formulated in the next theorem, which is the main result of this paper.

Theorem 1. *Let* $\{w_k : k = 1, 2, \dots\}$ *be a nondecreasing sequence of positive numbers for which*

$$\lim_{k\to\infty} w_k/\log k = 0. \tag{2.1}$$

Then there exist a sign type ONS $\{\phi_k(x)\}$ *of step functions on the interval* $(0, 1)$ *with*

$$\int_0^1 \phi_k(x)dx = 0 \qquad (k = 1, 2, \dots), \tag{2.2}$$

and a sequence $\{a_k\}$ *of real numbers such that*

$$\sum_{k=1}^{\infty} a_k^2 w_k^2 < \infty, \tag{2.3}$$

while the maximal partial sum $S_*(x)$ *does not belong to* $L^p(0, 1)$ *for any* $p > 0$.

We recall that a function $\phi(x)$ defined on the interval $(0, 1)$ is said to be a step function if there exists a partition of $(0, 1)$ into a finite number of disjoint subintervals such that $\phi(x)$ assumes a constant value on each of these subintervals.

Problem 1. We do not know whether Th. 1 can be improved in such a way that in its conclusion the a.e. convergence of the orthogonal series (1.2) in question can also be stated.

Problem 2. Is it true or not that given an arbitrary sequence $a_1 \geq a_2 \geq \geq a_3 \geq \dots$ with $a_k \to 0$ as $k \to \infty$ such that condition (1.1) is not satisfied, there exists a sign type (or only uniformly bounded) ONS $\{\phi_k(x)\}$ on $(0, 1)$ such that $S_*(x)$ does not belong to $L^2(0, 1)$; or even more, $S_*(x)$ does not belong to $L^p(0, 1)$ for any $p > 0$?

Problem 3. We do not know whether there exists a sign type (or only uniformly bounded) ONS $\{\phi_k(x)\}$ on (0,1) which is a convergence system, but $S_*(x)$ does not belong to $L^2(0, 1)$ for some $\{a_k\} \in \ell^2$.

In the proof of Th. 1, we rely on an inequality proved by Kashin [1] (see also [2, p. 258]).

Lemma 1. *For each* $m = 1, 2, \dots$ *there exists a sign type* ONS $\{\phi_k(x)\}$ *of step functions on the interval* $(0, 1)$ *such that*

$$\text{mes}\{x \in (0, 1) : \max_{1\leq n\leq 2m^2} \Big|\sum_{k=1}^{n} \phi_k(x)\Big| \geq C_1 m \log m\} \geq C_2.$$

By "mes" we denote the Lebesgue measure on the real line.

We make two comments to Lemma 1.

(i) It follows immediately that

$$(2.4)\qquad \text{mes}\{x \in (0,1) : \max_{1\le n_1\le n_2\le 2m^2}\Big|\sum_{k=n_1}^{n_2} \phi_k(x)\Big| \ge \frac{1}{2}C_1 m \log m\} \ge C_2.$$

(ii) We may assume that the ONS $\{\phi_k(x)\}$ in Lemma 1 satisfies condition (2.2), too. Otherwise, we could consider the functions

$$\tilde{\phi}_k(x) := \begin{cases} \phi_k(2x) & \text{if } x \in (0,1/2) \\ -\phi_k(2x-1) & \text{if } x \in (1/2,1) \end{cases}$$

instead of $\phi_k(x)$. It is plain that $\{\tilde{\phi}_k(x)\}$ is also an ONS satisfying conditions (1.4), (2.4) as well as (2.2).

After these preliminaries, we fix a sequence $\{m_s : s = 1, 2, \dots\}$ of positive integers with the following properties:

$$(2.5)\qquad N_s := 2\sum_{q=1}^{s} m_q^2 \le 2m_{s+1}^2,$$

$$(2.6)\qquad w_k^2/\log^2 k \le 2^{-3s} \quad \text{if} \quad k > N_s \quad (s = 1, 2, \dots).$$

This choice is possible due to (2.1).

For each $s \ge 1$, Lemma 1 guarantees the existence of an ONS $\{\phi_k^{(s)}(x) : k = 1, 2, \dots, 2m_s^2\}$ of step functions satisfying (1.4), (2.2), and

$$(2.7)\qquad \text{mes}(E_s) \ge C_2,$$

where

$$E_s := \{x \in (0,1) : \max_{1\le n_1\le n_2\le 2m_s^2}\Big|\sum_{k=n_1}^{n_2} \phi_k^{(s)}(x)\Big| \ge \frac{1}{2}C_1 m_s \log m_s\}.$$

Clearly, each E_s is a simple set, that is, E_s consists of a finite number of disjoint subintervals of (0,1).

By induction, we shall define an ONS $\{\phi_k(x) : k = 1, 2, \dots\}$ of step functions satisfying (1.4) and (2.2), and a sequence $\{H_s : s = 1, 2, \dots\}$ of simple sets of (0,1) such that

$$(2.8)\qquad \text{mes}(H_s) = \text{mes}(E_s) \ge C_2 \quad (s = 1, 2, \dots),$$

$$(2.9)\qquad \max_{N_{s-1}<n_1\le n_2\le N_s}\Big|\sum_{k=n_1}^{n_2} \phi_k(x)\Big| \ge \frac{1}{2}C_1 m_s \log m_s \quad \text{if} \quad x \in H_s,$$

where $N_0 := 0$. From the construction it turns out that the sets $\{H_s : s = 1, 2, \dots\}$ are actually stochastically independent, but we do not use this property in the sequel.

For $s = 1$, we set

$$\phi_k(x) := \phi_k^{(1)}(x) \quad (k = 1, 2, \dots, N_1 := 2m_1^2) \quad \text{and} \quad H_1 := E_1.$$

Then (2.8) and (2.9) are obviously satisfied.

Now, let s_0 be a positive integer and assume that the step functions $\phi_k(x)$ $(k = 1, 2, \ldots, N_{s_0})$ and the simple sets H_s $(s = 1, 2, \ldots, s_0)$ have been defined in such a way that these functions are orthonormal on (0,1) and conditions (1.4), (2.2), (2.8) and (2.9) are satisfied for $s = 1, 2, \ldots, s_0$. We take a partition $\{I_r : r = 1, 2, \ldots, \rho\}$ of the interval (0,1) into disjoint subintervals such that each function $\phi_k(x)$ $(k = 1, 2, \ldots, N_{s_0})$ assumes a constant value on each subinterval I_r $(r = 1, 2, \ldots, \rho)$.

We shall use the following notations. Given a function $\phi(x)$ defined on (0,1), a subinterval $I := (a, b)$ and a subset H of (0,1), we define

$$\phi(I; x) := \begin{cases} \phi(\frac{x-a}{b-a}) & \text{if } x \in I \\ 0 & \text{othervise} \end{cases}$$

and define $H(I)$ to be the set into which H is carried over by the linear transformation $y := (b - a)x + a$.

Now, we set

$$\phi_{N_{s_0}+k}(x) := \sum_{r=1}^{\rho} \phi_k^{(s_0+1)}(I_r; x) \quad (k = 1, 2, \ldots, 2m_{s_0+1}^2),$$

$$H_{s_0+1} := \bigcup_{r=1}^{\rho} E_{s_0+1}(I_r).$$

It is plain that the $\phi_k(x)$ $(k = 1, 2, \ldots, N_{s_0+1})$ are step functions, orthonormal on (0,1), and conditions (1.4), (2.2), (2.8) and (2.9) are satisfied for $s = s_0 + 1$, too.

Finally, we put

$$\text{(2.10)} \qquad a_k := \frac{2^s}{m_s \log m_s} \quad \text{if} \quad N_{s-1} < k \le N_s \quad (s = 1, 2, \ldots).$$

First, we check the fulfillment of (2.3). Indeed, by (2.5) and (2.6), we have

$$\sum_{k=1}^{\infty} a_k^2 w_k^2 = \sum_{k=1}^{N_1} a_k^2 w_k^2 + \sum_{s=1}^{\infty} \sum_{k=N_s+1}^{N_{s+1}} \frac{2^{2s+2}}{m_{s+1}^2 \log^2 m_{s+1}} w_k^2 \le$$

$$\le \sum_{k=1}^{N_1} a_k^2 w_k^2 + 16 \sum_{s=1}^{\infty} 2^{-s} < \infty.$$

Second, by (2.8) - (2.10), we have

$$\max_{N_s < n_1 \le n_2 \le N_{s+1}} \Big| \sum_{k=n_1}^{n_2} a_k \phi_k(x) \Big| \ge C_1 2^s \quad \text{if} \quad x \in H_{s+1},$$

whence, for any $p > 0$, it follows that

$$\int_0^1 \Big(\max_{1\le n\le N_{s+1}} \Big|\sum_{k=1}^{n} a_k\phi_k(x)\Big|\Big)^p dx \ge$$

$$\ge 2^{-p} \int_0^1 \Big(\max_{1\le n_1\le n_2\le N_{s+1}} \Big|\sum_{k=n_1}^{n_2} a_k\phi_k(x)\Big|\Big)^p dx \ge$$

$$\ge 2^{-p} \int_{H_{s+1}} \Big(\max_{N_s< n_1\le n_2\le N_{s+1}} \Big|\sum_{k=n_1}^{n_2} a_k\phi_k(x)\Big|\Big)^p dx \ge$$

$$\ge 4^{-p}C_1^p 2^{p(s+1)}\mathrm{mes}(H_{s+1}) \ge 4^{-p}C_1^p C_2 2^{p(s+1)} \quad (s=1,2,\dots).$$

This proves that $S_*(x)$ does not belong to $L^p(0,1)$ for any $p>0$. ◊

3. Uniformly bounded ONS

According to the Menshov-Paley theorem (see, e.g. [8, Vol. 2, p. 189]) if $\{\phi_k(x)\}$ is a uniformly bounded ONS on a positive measure space (X,μ), say

$$|\phi_k(x)|\le B \qquad (k=1,2,\dots;x\in X),$$

and for some $p>2$ we have

$$\mathfrak{L}_p[a] := \Big(\sum_{k=1}^{\infty} |a_k|^p k^{p-2}\Big)^{1/p} < \infty, \tag{3.1}$$

then the orthogonal series (1.2) converges a.e. (this is an immediate consequence of the fact that, by Hölder's inequality, (3.1) implies (1.1)), the maximal partial sum $S_*(x)$ belongs to $L^p(X,\mu)$, and

$$\|S_*\|_p := \Big\{\int_X S_*^p(x)d\mu(x)\Big\}^{1/p} \le C_p B^{(p-2)/p}\mathfrak{L}_p[a], \tag{3.2}$$

where the value of the constant C_p depends only on p, but not on $\{\phi_k(x)\}$ and $\{a_k\}$.

Given a sequence $\{a_k\}$ of real numbers with $a_k\to 0$ as $k\to\infty$, denote by $\{a_k^*\}$ the sequence $|a_1|$, $|a_2|,\dots$ rearranged in a descending order of magnitude, while deleting the terms a_k equal to 0. In case several $|a_k|$ are equal, we rearrange them in the order of increasing index k. Now, (3.2) does hold even if $\mathfrak{L}_p[a^*]$ is substituted for $\mathfrak{L}_p[a]$ on its right-hand side. This improvement is significant, since

$$\mathfrak{L}_p[a^*]\le C_p\|a\|_{p'} := C_p\Big(\sum_{k=1}^{\infty}|a_k|^{p'}\Big)^{1/p'}, \quad \frac{1}{p}+\frac{1}{p'}=1;$$

which is not true in general with $\mathfrak{L}_p[a]$ instead of $\mathfrak{L}_p[a^*]$.

It is well known that in the particular case where $a_1 \geq a_2 \geq \dots$ and $a_k \to 0$ as $k \to \infty$, the cosine series

$$\sum_{k=1}^{\infty} a_k \cos kx =: f(x) \tag{3.3}$$

converges, except possibly $x = 0 \bmod 2\pi$. Hardy and Littlewood (see, e.g. [8, Vol. 2, p. 129]) proved that the inequality converse to (3.2) holds for every $p > 1$, that is,

$$\mathfrak{L}_p[a] \leq C_p \|f\|_p.$$

Clearly, we have $\|f\|_p \leq \|S_*\|_p$ for every $p > 0$. We note that an analogous inequality holds for sine series, too.

Hence it follows immediately that if condition (1.1) is satisfied, then the integrability statement $S_*(x) \in L^2(X, \mu)$ expressed in (1.3) is the best possible in the sense that L^2 cannot be replaced by L^p for any $p > 2$. In fact, consider the nonincreasing sequence

$$a_k := (s+1)^{-2} 2^{-s/2} \quad \text{if} \quad 2^s < k \leq 2^{s+1} \quad (s = 0, 1, \dots).$$

Then

$$\sum_{k=1}^{\infty} a_k^2 \log^2 k \leq \sum_{s=0}^{\infty} \sum_{k=2^s+1}^{2^{s+1}} a_k^2 (s+1)^2 = \sum_{s=0}^{\infty} (s+1)^{-2} < \infty,$$

while for every $p > 2$,

$$\sum_{k=1}^{\infty} a_k^p k^{p-2} \geq \sum_{s=0}^{\infty} \sum_{k=2^s+1}^{2^{s+1}} a_k^p 2^{s(p-2)} = \sum_{s=0}^{\infty} \frac{2^{s(-1+p/2)}}{(s+1)^{2p}} = \infty.$$

Thus, in this special case, the maximal partial sum $S_*(x)$ of series (3.3) belongs to $L^2(0, 2\pi)$, but does not belong to $L^p(0, 2\pi)$ for any $p > 2$.

4. Nonuniformly bounded ONS

We shall see that if an ONS $\{\phi_k(x)\}$ is such that each $\phi_k(x)$ is bounded in $x \in X$, but they are not uniformly bounded in $k = 1, 2, \dots$, then we cannot expect any reasonable condition in order to guarantee that $S_*(x)$ belongs to $L^p(X, \mu)$ for some $p > 2$.

Theorem 2. *Let $\{a_k\}$ be an arbitrary sequence of real numbers for which*

$$\sum_{k=n}^{\infty} a_k^2 > 0 \quad (n = 1, 2, \dots). \tag{4.1}$$

Then there exists an ONS $\{\phi_k(x)\}$ of step functions on the interval $(0, 1)$ such that $S_*(x)$ does not belong to $L^p(0, 1)$ for any $p > 2$.

Clearly, condition (4.1) is equivalent to the fact that the sequence $\{a_k\}$ contains infinitely many nonzero terms. So, in case $\sum |a_k| < \infty$ the orthogonal series (1.2) in question converges absolutely a.e.; nevertheless, the maximal partial sum $S_*(x)$ may not belong to $L^p(0,1)$ for any $p > 2$.

In the proof, we shall make use of the norm introduced by the second named author (see [5] and [6]) in the study of the a.e. convergence of general orthogonal series. To this end, denote by Ω the class of all (not necessarily bounded) ONS $\{\phi_k(x)\}$ on the interval (0,1). Given a sequence $a = \{a_k\}$ of real numbers, define

$$\|a\| := \sup_{\Omega}\left\{\int_0^1 \left(\sup_{n\ge1}\left|\sum_{k=1}^{n} a_k\phi_k(x)\right|\right)^2 dx\right\}^{1/2},$$

which may be infinite. It is proved in [6] that the orthogonal series (1.2) converges a.e. for each $\{\phi_k(x)\} \in \Omega$ if and only if $\|a\| < \infty$, in which case $S_*(x)$ belongs to $L^2(0,1)$

For $1 \le M \le N < \infty$, we set

$$a(M,N) := \{0,\dots,0,\ a_M,\ a_{M+1},\dots,\ a_N,\ 0,0,\dots\}.$$

By orthogonality, it is clear that

$$\sum_{k=1}^{N} a_k^2 \le \|a(1,N)\|^2 \quad (N = 1,2,\dots). \tag{4.2}$$

The next lemma was proved by the second named author [5].

Lemma 2. *Let $a = \{a_k\}$ be an arbitrary sequence of real numbers and N a positive integer. Then there exists a finite* ONS $\{\phi_k(x) : k = 1,2,\dots,N\}$ *of step functions on* $(0,1)$ *such that*

$$\int_0^1 \left(\max_{1\le n\le N}\left|\sum_{k=1}^{n} a_k\phi_k(x)\right|\right)^2 dx \ge \frac{1}{16}\|a(1,N)\|^2.$$

Proof of Theorem 2. From (4.1) and (4.2) it follows that there exists a sequence $(1 =)N_1 < N_2 < \dots < N_s < \dots$ of integers such that

$$\|a(N_s, N_{s+1}-1)\| > 0 \quad (s = 1,2,\dots).$$

Let $\{\ell_s > 1\}$ be a sequence of positive integers for which

$$\sum_{s=1}^{\infty} \ell_s^{-2}\|a(N_s, N_{s+1}-1)\|^2 \le 1, \tag{4.3}$$

and such that for each positive integer r,

$$\ell_s^{1/r}\|(N_s, N_{s+1}-1)\|^2 \ge 1 \quad \text{if} \quad s \ge M_r, \tag{4.4}$$

where M_r depends only on r.

By (4.3), we can select a sequence $\{I_s\}$ of disjoint subintervals of (0,1) such that

$$\text{mes}(I_s) = \ell_s^{-2}\|a(N_s, N_{s+1}-1)\|^2 \quad (s = 1, 2, \dots). \tag{4.5}$$

For each $s \geq 1$, we apply Lemma 2. As a result, we obtain ONS $\{\phi_k^{(s)}(x) : k = N_s, N_s+1, \dots, N_{s+1}-1\}$ of step functions on (0,1) such that

$$\begin{aligned}\int_0^1 \Big(\max_{N_s \leq n < N_{s+1}} \Big|\sum_{k=N_s}^{n} a_k \phi_k^{(s)}(x)\Big|\Big)^2 dx \geq \\ \geq \frac{1}{16}\|a(N_s, N_{s+1}-1)\|^2 \quad (s = 1, 2, \dots).\end{aligned} \tag{4.6}$$

Setting $\phi_k(x) := \ell_s\|a(N_s, N_{s+1}-1)\|^{-1}\phi_k^{(s)}(I_s; x)$ $(k = N_s, \dots, N_{s+1} - -1;\ \ s = 1, 2, \dots)$, it is plain that $\{\phi_k(x) : k = 1, 2, \dots\}$ is an ONS of step functions on (0,1).

Let $p > 2$ be given. Since for an arbitrary function $\phi(x)$ defined on (0,1) and a subinterval $I \subseteq (0, 1)$, we have

$$\int_0^1 |\phi(I; x)|^p dx = \text{mes}(I) \int_0^1 |\phi(x)|^p dx,$$

we can write the following:

(4.7)

$$\begin{aligned}\int_0^1 \Big(\sup_{n\geq 1}\Big|\sum_{k=1}^{n} a_k\phi_k(x)\Big|^p dx = \sum_{s=1}^{\infty}\int_0^1 \Big(\max_{N_s \leq n < N_{s+1}}\Big|\sum_{k=N_s}^{n} a_k\phi_k(x)\Big|\Big)^p dx = \\ = \sum_{s=1}^{\infty} \ell_s^p \|a(N_s, N_{s+1}-1)\|^{-p} \int_0^1 \Big(\max_{N_s \leq n < N_{s+1}}\Big|\sum_{k=N_s}^{n} a_k\phi_k^{(s)}(I_s; x)\Big|\Big)^p dx = \\ = \sum_{s=1}^{\infty} \ell_s^p \|a(N_s, N_{s+1}-1)\|^{-p}\text{mes}(I_s) \int_0^1 \Big(\max_{N_s \leq n < N_{s+1}}\Big|\sum_{k=N_s}^{n} a_k\phi_k^{(s)}(x)\Big|\Big)^p dx.\end{aligned}$$

By Hölder's inequality (we recall that $p > 2$) and (4.6), we have

$$\begin{aligned}\Big\{\int_0^1 \Big(\max_{N_s \leq n < N_{s+1}}\Big|\sum_{k=N_s}^{n} a_k\phi_k^{(s)}(x)\Big|\Big)^p dx\Big\}^{1/p} \geq \\ \geq \Big\{\int_0^1 \Big(\max_{N_s \leq n < N_{s+1}}\Big|\sum_{k=N_s}^{n} a_k\phi_k^{(s)}(x)\Big|\Big)^2 dx\Big\}^{1/2} \geq \frac{1}{16}\|a(N_s, N_{s+1}-1)\|\end{aligned}$$

for all $s = 1, 2, \dots$. Substituting this on the right-hand side of (4.7), while taking (4.5) into account yields

$$\int_0^1 \Big(\sup_{n\ge 1}\Big|\sum_{k=1}^n a_k\phi_k(x)\Big|\Big)^p dx \ge$$

$$\ge 16^{-p}\sum_{s=1}^{\infty}(\ell_s\|a(N_s,N_{s+1}-1)\|^{-1})^{p-2}\|a(N_s,N_{s+1}-1)\|^p = \tag{4.8}$$

$$= 16^{-p}\sum_{s=1}^{\infty}\ell_s^{p-2}\|a(N_s,N_{s+1}-1)\|^2.$$

If we choose the integer r so large that $p-2\ge 1/r$, then combining (4.4) and (4.8) gives

$$\int_0^1 \Big(\sup_{n\ge 1}\Big|\sum_{k=1}^n a_k\phi_k(x)\Big|\Big)^p dx \ge 16^{-p}\sum_{s=1}^{\infty}\ell_s^{1/r}\|a(N_s,N_{s+1}-1)\|^2 = \infty.\Diamond$$

5. Concluding remarks.

Let $\{\phi_k(x)\}$ be a sequence whose members are functions (random variables) defined on the interval (0,1) and stochastically (totally) independent with zero mean (i.e., condition (2.2) is satisfied). Clearly, then $\{\phi_k(x)\}$ is an ONS on (0,1). Marcinkiewicz and Zygmund [3] proved that in this case for every $p>1$, we have

$$C_{1p}\Big\|\Big(\sum_{k=1}^{\infty}\phi_k^2\Big)^{1/2}\Big\|_p \le \Big\|\sum_{k=1}^{\infty}\phi_k\Big\|_p \le C_{2p}\Big\|\Big(\sum_{k=1}^{\infty}\phi_k^2\Big)^{1/2}\Big\|_p, \tag{5.1}$$

where C_{1p} and C_{2p} are positive constants, whose values depend only on p. Furthermore, they also proved that in this case for every $p>1$, we have

$$\|S_*\|_p \le 2^{1/p}\frac{p}{p-1}\Big\|\sum_{k=1}^{\infty}\phi_k\Big\|_p. \tag{5.2}$$

Now, we consider the wellknown Rademacher ONS $\{r_k(x)\}$, as a special case. From (5.1) and (5.2) it follows immediately that if $\sum a_k^2 <$ $< \infty$, then the inequalities

$$C_{1p}\Big(\sum_{k=1}^{\infty}a_k^2\Big)^{1/2} \le \Big\|\sup_{n\ge 1}\Big|\sum_{k=1}^n a_k r_k(x)\Big|\Big\|_p \le \tilde{C}_{2p}\Big(\sum_{k=1}^{\infty}a_k^2\Big)^{1/2} \tag{5.3}$$

hold for every $p>1$, where $\tilde{C}_{2p}$ is also a positive constant depending only on p. In particular, the maximal partial sum $S_*(x)$ of the Rademacher series $\sum a_k r_k(x)$ is a bounded operator from ℓ^2 to $L^2(0,1)$ (both from above and from below). It is known that inequalities (5.3) hold true even for every $p>0$ (see, e.g. [8, Vol. 1, p. 213]).

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