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$\mathbb{L}^p$ NORM CONVERGENCE OF RATIONAL OPERATORS ON THE UNIT CIRCLE

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Abstract: This paper provides a generalization of certain classical $\mathbb{L}^p$ norm convergence and summation theorems of the partial sums of Fourier series to the case where the underlying orthonormal basis is not the trigonometric one, but a rational generalization which contains the trigonometric one as a special case. It is introduced a rational interpolation operator on nodes given on the unit circle. By using a generalization of the Marcinkiewicz classical $\mathbb{L}^p$ norm convergence theorems for triginometric interpolation $\mathbb{L}^p$ norm convergence is proved for the discrete rational operators, too.

1. Introduction

In the area of applied mathematics a fundamental idea is that of expressing solutions by expanding them in terms of orthogonal basis functions, e.g., the classical Fourier analysis, classical orthogonal polynomials and solutions of self–adjoint operator equations in terms of the orthogonal eigenfunctions of the operator. More recently, for signal processing and system theoretic problems there has been an explosion

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of interest in the development and use of wide class of new orthogonal bases.

Over the last years a general theory has been developed for the construction and analysis of rational orthonormal basis functions, often called generalized orthonormal basis functions in the engineering literature, for the class of stable linear systems. These basis are parameterized in terms of pre–specified poles that makes it possible to incorporate a priori information in the model structure [7], [5], [6], [12], [11], [10]. These recently developed basis functions have been shown to have attractive properties in several respects. The use as linear model parametrizations in system identification problems has been shown to be attractive, due to the fact that smartly chosen basis functions can provide a fast rate of convergence of the corresponding series expansion, leading to linear model parametrizations with a limited number of parameters.

These investigations motivate the interest in the examination of the approximation properties of the rational orthonormal systems generated by a given set of poles. A generic example of such a system is the so called Takenaka–Malmquist system, see [2], [13]. These basis can be viewed as an extension of the trigonometric system on the unit circle, that corresponds to the special choice when all of the poles are located at the origin. This paper provides a generalization of certain classical $\mathbb{L}^p$ norm convergence and summation theorems of the partial sums of Fourier series to the case where the underlying orthonormal basis is a rational one that contains the trigonometric basis as a special case.

This paper provides a generalization of the Marcinkiewicz classical $\mathbb{L}^p$ norm convergence theorem of the trigonometric interpolation on equidistant nodes on the unit circle, see [14], to the rational interpolation process generated by the case where the underlying orthonormal basis is a rational one that contains the trigonometric basis as a special case.

The structure of the presentation is the following: after fixing the basic notations and introducing the rational orthonormal functions some known facts are recalled about the reproducing kernels of the subspaces generated by these functions. An extension is given from $\mathbb{H}^2$, i.e, the Hilbert space of square integrable functions on the unit circle with analytic extension on the unit disc, to $\mathbb{L}^2$, the space of square integrable functions on the unit circle. This is followed by a section that proves the uniform boundedness in $\mathbb{L}^p$ norm of the partial sum operators and the $\mathbb{L}^p$ norm convergence of these sums. The next section gives an extension of the classical summation properties to the rational orthonormal basis situation, when the basis is generated by a periodic set of poles. The second part of the paper introduces a discrete rational orthonormal system on the unit circle and provides a theorem for $\mathbb{L}^p$ norm convergence of these rational interpolation operators.

2. Basic notations

Denote by $\mathbb{C}$ the set of complex numbers and let $\mathbb{Z}$ be the set of integers. The open unit disc and its boundary will be denoted by $\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\}$. By $\mathbb{L}^p$ will be denoted the classical $\mathbb{L}^p(\mathbb{T})$ Banach space endowed with the norm $||f||_p :$ $:= (\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it})|^p dt)^{\frac{1}{p}}$, and $\mathbb{H}^2$ will be the Hardy space of square integrable functions on $\mathbb{T}$ with analytic continuation on the unit disc. Its orthogonal complement in $\mathbb{L}^2$ will be denoted by $\mathbb{H}^{2\perp}$. The scalar product considered is the usual one, i.e., $< f, g > := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\overline{g}(e^{it})dt$. Throughout this paper $z := e^{it}$ will denote a complex number from $\mathbb{T}$.

Let B_n be a finite Blaschke product of order $n \in \mathbb{N} := \{1, 2, \dots\}$ written under the form

$$B_n := \prod_{j=1}^{n} b_j, \quad b_j(z) := \frac{z - \alpha_j}{1 - \overline{\alpha}_j z}$$

where $|\alpha_j| < 1$, $(j = 1, 2, \dots, n)$ are given complex numbers. Let us consider the subspace $\mathbf{H}(B_n) := \mathbb{H}^2 \ominus B_n\mathbb{H}^2$, i.e, $\mathbf{H}(B_n)$ is the orthogonal complement of $B_n\mathbb{H}^2$ in $\mathbb{H}^2$.

Consider the set of functions ϕ_j associated to the set of zeros $\{\alpha_j \,|\, j = 1, \dots, n\}$ of B_n defined by

$$\phi_1(z) := \frac{d_1}{1 - \overline{\alpha}_1 z}; \phi_j(z) := \frac{d_j}{1 - \overline{\alpha}_j z}\prod_{i=1}^{j-1} b_{\alpha_i}(z), \quad 1 < j \leq n,$$

where $d_j := \sqrt{1 - |\alpha_j|^2}$.

It is clear that $\phi_j \in \mathbf{H}(B_n)$, $\quad ||\phi_j||_2 = 1$ and they form an orthonormal basis of $\mathbf{H}(B_n)$, the so-called Takenaka–Malmquist system.

As a starting point, let us mention the fact that for a finite Blaschke product B_n of order n, there exist a monotone increasing, invertable and differentiable function $\beta_{(n)}(t)$ mapping the interval $[-\pi, \pi)$ onto itself, see [9], so that

$$B_n(e^{it}) = e^{in\beta_{(n)}(t)}.$$

Denote by $\gamma_{(n)}(t)$ the inverse function $\beta_{(n)}^{-1}(t)$.

Let us denote the phase function of a single term by $\beta_k(t)$, i.e., $b_{\alpha_k}(e^{it}) = e^{i\beta_k(t)}$, then the function $\beta_{(n)}(t)$ can be expressed as

$$\beta_{(n)}(t) := \frac{1}{n}\sum_{k=1}^{n} \beta_k(t).$$

For the derivatives one has, see [11]

$$\beta_k'(t) = \frac{1-|\alpha_k|^2}{|1-\overline{\alpha}_k e^{it}|}.$$

Since $1-|\alpha_k| \leq |1-\overline{\alpha}_k e^{it}| \leq 1+|\alpha_k|$, one can obtain the bounds

$$\frac{1-|\alpha_k|}{1+|\alpha_k|} \leq \beta_k'(t) \leq \frac{1+|\alpha_k|}{1-|\alpha_k|},$$

and hence

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1-|\alpha_k|}{1+|\alpha_k|} \leq \beta_{(n)}'(t) \leq \frac{1}{n}\sum_{k=1}^{n}\frac{1+|\alpha_k|}{1-|\alpha_k|}.$$

It follows that the derivative of the inverse is bounded by

$$\frac{n}{\sum_{k=1}^{n}\frac{1+|\alpha_k|}{1-|\alpha_k|}} \leq \gamma_{(n)}'(t) \leq \frac{n}{\sum_{k=1}^{n}\frac{1-|\alpha_k|}{1+|\alpha_k|}}.$$

3. Reproducing kernels for $\mathbf{H}(B_n)$, extensions to $\mathbb{L}^2$

In this section some basic facts will be summarized about the Takenaka–Malmquist system and a possible extension of this basis will be presented to $\mathbb{L}^2$. Let us recall the completeness property of this system in the $\mathbb{H}^p$ spaces, i.e.:

Theorem 3.1. *The system $\{\phi_k \mid k \in \mathbb{N}\}$ is complete in $\mathbb{H}^p$ if and only if*

$$\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty.$$

Proof. Let us consider that $\sum_{k=1}^{\infty}(1-|\alpha_k|)<\infty$. Then $f(z) = \prod_{k=1}^{\infty}\frac{\alpha_k - z}{1-\overline{\alpha}_k z}$ is an inner function in $\mathbb{H}^p$, see [4], pp. 64, and clearly $< f, \phi_k >= 0$ for all $k \in \mathbb{N}$ that contradicts the completeness of the system.

For the only if part let us suppose that the system is not complete, i.e., there exists a nonzero function $f \in \mathbb{H}^p$ such that $< f, \phi_k >= 0$ for every $k \in \mathbb{N}$. By the Cauchy theorem follows that f has as zeros the set $\{\alpha_k\}$. From it follows that $\sum_{k=1}^{\infty}(1-|\alpha_k|)<\infty$, see [3], pp. 53, which is a contradiction. That proves the assertion. ◊

The reproducing kernel $K(z,\mu)$, $(z,\mu \in \mathbb{T})$ of a closed subspace $V \subset \mathbb{L}^2$ is defined by

- for every μ the function $K(z,\mu)$ belongs to V,
- the reproducing property, i.e., for all $\mu \in \mathbb{T}$ and every

  $$f \in V\, f(\mu) =< f(z), K(z,\mu) >_z, \quad (\mu \in \mathbb{T}),$$

  where the subscript z by the scalar product indicates that the scalar product applies to functions of z.

If an orthonormal basis is considered in the finite dimensional subspace V say $\{\varphi_j(z)\,|\, j = 1,\ldots,n\}$, where $n = dimV$, then the reproducing kernel is given by $K_n(z,\mu) = \sum_{k=1}^{n}\varphi_k(z)\overline{\varphi}_k(\mu)$, $z,\mu \in \mathbb{T}$, and it is independent of the choice of the orthonormal system, see [1]. Applied this to the subspace $\mathbf{H}(B_n)$ one can obtain $K_n(z,\mu) = \sum_{k=1}^{n}\phi_k(z)\overline{\phi}_k(\mu)$, as a reproducing kernel.

One has the following Christoffel–Darboux formula [2], pp. 320:

Lemma 3.2.

$$K_n(z,\mu) := \sum_{k=1}^{n}\phi_k(z)\overline{\phi}_k(\mu) = \frac{1 - B_n(z)\overline{B}_n(\mu)}{1 - z\overline{\mu}}.$$

Proof. For the sake of completeness a short elementary proof is given below. By direct computation one has $\frac{1-b_k(z)\overline{b}_k(\mu)}{1-z\overline{\mu}} = \frac{1-|\alpha_k|^2}{(1-\overline{\alpha}_k z)(1-\alpha_k\overline{\mu})}$, i.e., by using the definition of the functions ϕ_k, it follows

$$\phi_k(z)\overline{\phi}_k(\mu) = \frac{1 - b_k(z)\overline{b}_k(\mu)}{1 - z\overline{\mu}}\prod_{j=1}^{k-1} b_j(z)\overline{b}_j(\mu) =$$

$$= \frac{B_{k-1}(z)\overline{B}_{k-1}(\mu) - B_k(z)\overline{B}_k(\mu)}{1 - z\overline{\mu}}.$$

By summing up this formula results our assertion. ◊

There are a lot of possibilities to extend an orthonormal system from $\mathbb{H}^2$ to $\mathbb{L}^2$. Let us consider the map $U : \mathbb{H}^2 \to \mathbb{H}^{2\perp}$ $Uf(z) = \overline{zf(z)}$. Then the system defined by $\phi_{-k} = U\phi_k$, $(k \in \mathbb{N})$ forms an orthonormal system in $\mathbb{H}^{2\perp}$. A finite set $\{\phi_{-k} \mid k = 1, \ldots, n\}$ that corresponds to the system of poles $\{\frac{1}{\alpha_k}\}$ span the subspace $\mathbf{H}(\overline{B}_n) = \mathbb{H}^{2\perp} \ominus \overline{B}_n \mathbb{H}^{2\perp}$. Let us mention that $\mathbf{H}(\overline{B}_n) = \overline{B}_n \mathbf{H}(B_n)$ and the system $\{\phi_{*k} \mid \phi_{*k} = \overline{B}_n \phi_k\}$ is an orthonormal basis in $\mathbf{H}(\overline{B}_n)$.

Using the formula of the reproducing kernel of $\mathbf{H}(B_n)$ one can obtain the reproducing kernel of $\mathbf{H}(\overline{B}_n) \oplus \mathbf{H}(B_n)$, i.e.,

$$s_n^{\alpha}(z, \mu) := \sum_{k=-n, k\neq 0}^{n} \phi_k(z)\overline{\phi}_k(\mu) = (z\overline{\mu})^{-\frac{1}{2}} \frac{B_n(z)\overline{B}_n(\mu) - B_n(\mu)\overline{B}_n(z)}{(z\overline{\mu})^{\frac{1}{2}} - (\mu\overline{z})^{\frac{1}{2}}},$$

$(z, \mu \in \mathbb{T})$. Let $z = e^{it}$ and $\mu = e^{i\tau}$, then one has

$$S_n^{\alpha}(t, \tau) := s_n^{\alpha}(e^{it}, e^{i\tau}) = e^{-i\frac{t-\tau}{2}} \frac{\sin n(\beta_{(n)}(t) - \beta_{(n)}(\tau))}{\sin(\frac{t-\tau}{2})},$$

where $t, \tau \in [-\pi, \pi)$.

If one consider the system generated by the set of poles

$$\{\frac{1}{\alpha_k},\ 0,\ \alpha_k \mid k = 1, \ldots, n\},$$

i.e., the subspace $\mathbf{H}(\overline{B}_n) \oplus \mathbf{H}(0) \oplus z\mathbf{H}(B_n)$, then one can obtain

$$d_n^{(\alpha)}(z, \mu) = \frac{(z\overline{\mu})^{\frac{1}{2}} B_n(z)\overline{B}_n(\mu) - (\mu\overline{z})^{\frac{1}{2}} B_n(\mu)\overline{B}_n(z)}{(z\overline{\mu})^{\frac{1}{2}} - (\mu\overline{z})^{\frac{1}{2}}},$$

i.e.

$$D_n^{(\alpha)}(t, \tau) = \frac{\sin\left(n(\beta_{(n)}(t) - \beta_{(n)}(\tau)) + \frac{1}{2}(t - \tau)\right)}{\sin(\frac{t-\tau}{2})}.$$

This can be considered as a generalization of the Dirichlet kernel

$$D_n(t, \tau) = \frac{\sin(n + \frac{1}{2})(t - \tau)}{\sin(\frac{t-\tau}{2})}$$

that can be obtained when all of the poles are placed at the origin.

4. $\mathbb{L}^p$ norm convergence of the partial sums

This section gives a generalization of the classical $\mathbb{L}^p$ norm convergence theorem of the partial sums of the Fourier series to the partial sums in the rational orthonormal system $\{\phi_k \mid k \in \mathbb{Z}\}$ generated by the poles $\{\frac{1}{\alpha_k},\ \alpha_k \mid k \in \mathbb{N}\}$.

Let us denote the partial sums of the expansion of a function f in the orthonormal system $\{\phi_k \mid k \in \mathbb{Z}\}$ by $\Sigma_n f$ i.e., $\Sigma_n f =< f, S_n^{(\alpha)} >$.

Theorem 4.1. *For* $f \in \mathbb{L}^p$, $\quad 1 < p < \infty$ *one has*

$$||\Sigma_n f||_p \leq C||f||_p$$

and if $\sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty$ *then*

$$\lim_{n\to\infty} ||f - \Sigma_n f||_p = 0.$$

Proof. The proof follows classical lines in using the theorem of M. Riesz about the conjugate functions, i.e., for $f \in \mathbb{L}^p$, $\quad 1 < p < \infty$, $\quad \tilde{f} \in \mathbb{L}^p$ and $||\tilde{f}||_p \leq C_p||f||$. The conjugate of a function f is defined by

$$\tilde{f}(t) = PV\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-\tau)\operatorname{cotg}\frac{\tau}{2}d\tau,$$

where PV denotes the principal value of the integral. One has

$$\Sigma_n f(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)S_n^{(a)}(t,\tau))d\tau =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\sin n(\beta_{(n)}(t) - \beta_{(n)}(\tau))\operatorname{cotg}(\frac{t-\tau}{2})d\tau +$$

$$+\frac{i}{2\pi}\int_{-\pi}^{\pi} f(\tau)\sin n(\beta_{(n)}(t) - \beta_{(n)}(\tau))d\tau = F_1(t) + iF_2(t).$$

It is clear that $F_1 = \tilde{F}_2$. Now, by Jensen inequality one has $||F_2||_p \leq C||f||_p$ and by the M. Riesz theorem $||F_1||_p \leq CC_p||f||_p$. It follows that $||\Sigma_n f||_p \leq C||f||_p$, where C is a generic constant. Since for $\sum_{k=1}^{\infty}(1 - -|\alpha_k|) = \infty$ the system $\{\phi_k \mid k \in \mathbb{N}\}$ is complete in $\mathbb{L}^p$ by a consequence of the Banach–Steinhaus theorem one can obtain the second part of the assertion. $\Diamond$

5. Summation theorems for the periodic case

Let us consider the situation, when the set of poles that generate the orthonormal system is formed by a periodic repetition of the same finite sequence $\{\alpha_k \mid k = 1, \ldots, d\}$.

If one consider the finite Blaschke product $B(z) = \prod_{k=1}^{d} \frac{z-\alpha_k}{1-\overline{\alpha}_k z}$, and an orthonormal basis $\{\varphi_l \mid l = 1, \ldots, d\}$ in the subspace $\mathbf{H}(B)$, then the system $\phi_{l+kd} = \varphi_l B^k, \quad k \in \mathbb{Z}$ form an orthonormal basis of $\mathbb{L}^2$. For a proof based on the properties of the shift operator induced by the multiplication by the inner function B on $\mathbb{H}^2$ see [9] [10].

Let us consider for these type of systems the analogous of the Fejér summation, $\Phi_n = \frac{1}{n}\sum_{k=1}^n \Sigma_n$ i.e., the operator with the kernel

$$F_n^{(\alpha)}(t,\tau) = \frac{1}{n}\sum_{k=1}^n S_n^{(\alpha)}(t,\tau) \quad (t,\tau \in [-\pi,\pi)).$$

For the periodic case we have $S_n^{(\alpha)}(t,\tau) = e^{-i\frac{t-\tau}{2}}\frac{\sin nd(\beta(t)-\beta(\tau))}{\sin\frac{t-\tau}{2}}$ and using the fact that

$$\sum_{k=1}^n \sin kx = \frac{\sin^2\frac{nx}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}} + \frac{1}{2}\sin nx$$

one can obtain that

$$F_n^{(\alpha)}(t,\tau) =$$

$$= \frac{1}{n}e^{-i\frac{t-\tau}{2}}\Big(\frac{\sin^2\frac{nd}{2}(\beta(t)-\beta(\tau))}{\sin\frac{d}{2}(\beta(t)-\beta(\tau))}\frac{\sin\frac{d}{2}(\beta(t)-\beta(\tau))}{\sin\frac{t-\tau}{2}}\cos\frac{t-\tau}{2} +$$

$$+\frac{1}{2}\frac{\sin\frac{nd}{2}(\beta(t)-\beta(\tau))}{\sin\frac{t-\tau}{2}}\Big).$$

Theorem 5.1. *For all $f \in \mathbb{L}^\infty$ the following inequality holds*

$$||\Phi_n f||_\infty \le C||f||_\infty,$$

where $C > 0$ is an absolute constant and for all continuous f one has

$$\lim_{n\to\infty} ||f - \Phi_n f||_\infty = 0.$$

Proof. Proving the assertion is equivalent to show that the operators Φ_n are uniformly bounded, i.e., $|| < f, F_n^{(a)} > ||_\infty \le C||f||_\infty$. It is known from the classical Fourier series theory that the integrals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}dt$$

are uniformly bounded and that

$$A\log(n) \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right|dt \le B\log(n) \quad (n \in \mathbb{N})$$

with constants $A, B > 0$. Using the properties of the Lebesgue integral and of the β function one can show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2\frac{nd}{2}(\beta(t)-\beta(\tau))}{\sin^2\frac{d}{2}(\beta(t)-\beta(\tau))}dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\gamma'(t)dt$$

and

$$\left|\frac{\sin\frac{d}{2}(\beta(t)-\beta(\tau))}{\sin\frac{t-\tau}{2}}\right|=\left|\sum_{l=1}^{d}\varphi_l(e^{it})\overline{\varphi}_l(e^{i\tau})\right|\leq$$

$$\leq\sum_{l=1}^{d}\frac{1-|\alpha_l|^2}{|(1-\overline{\alpha}_l e^{it})(1-\alpha_l e^{-i\tau})|}\leq(\beta'(t)\beta'(\tau))^{\frac{1}{2}}\leq||\beta'||_\infty.$$

Putting these facts together follows that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_n^{(\alpha)}(t,\tau)|d\tau\right|\leq C_1||\beta'||_\infty||\gamma'||_\infty+C_2\frac{\log(n)}{n}\leq C,$$

i.e., $|<f,F_n^{(\alpha)}>|\leq C||f||_\infty$, which is the assertion of the theorem. $\Diamond$

One can introduce the generalization of the de La Valée Poussin operators as $V_n=\frac{1}{2n}\sum_{k=n+1}^{2n}\Sigma_k=2\Phi_{2n}-\Phi_n$, that has the same convergence properties.

Let us mention here that getting the square, [8], or the fourth power, [2], of the absolute value of $S_n^{(\alpha)}$ as kernels leads also to uniformly bounded operators, but these operators cannot be associated with simple summation processes.

6. $\mathbb{L}^p$ norm convergence of certain rational interpolation operators on the unit circle

Let us consider an orthonormal system $\{\phi_k\mid k=1,\ldots,n\}$ on the subspace $\mathbf{H}(B_n)$ generated by the finite Blaschke product B_n. Denote the set of equidistant nodes on the unit circle by $\mathbb{T}_n=\{\nu_k=\frac{2k\pi}{n}\mid k=0,\ldots,n-1\}$ and by $\mathbb{T}_n^\beta=\{\zeta_k=e^{i\gamma_k}\mid\gamma_k=\beta_{(n)}^{-1}(\frac{2k\pi}{n}),\ k=0,\ldots,n-1\}$. Let us define the discrete scalar product

$$[f,g]_n^\beta:=\frac{1}{n}\sum_{\zeta\in\mathbb{T}_n^\beta}f(\zeta)\overline{g}(\zeta).$$

Starting from the Christoffel–Darboux formula

$$K_n(e^{it},e^{i\tau})=\sum_{k=1}^{n}\phi_k(e^{it})\overline{\phi}_k(e^{i\tau})=\frac{1-e^{in(\beta_n(t)-\beta_n(\tau))}}{1-e^{i(t-\tau)}},$$

one can obtain $K_n(\zeta_l,\zeta_k)=0$ if $l\neq k$ and $K_n(\zeta_k,\zeta_k)=\frac{n}{\gamma_n'(\nu_k)}$.

From it follows that for the system

$$\left\{\tilde{\phi}_k = \frac{\phi_k}{\sqrt{\beta_n'}}\right\} \quad \text{onehas} \quad [\tilde{\phi}_k, \tilde{\phi}_l]_n^\beta = \delta_{k,l},$$

for $1 \leq k, l \leq n$, i.e., the discrete orthonormality holds.

One can derive the following interpolation operator:

$$(L_n f)(z) := \sum_{\zeta \in \mathbb{T}_n^\beta} \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)} f(\zeta),$$

where f is a continuous function on $\mathbb{T}$ and $z \in \mathbb{T}$. It is easy to see using the reproducing property of the kernel that

$$< L_n f, L_n g >= [f, g]_n^\beta.$$

From now on let us consider the case when the rational orthonormal system of functions is generated by a periodic set of poles, i.e, by the finite Blaschke product $B(z) = \prod_{k=1}^d \frac{z-\alpha_k}{1-\overline{\alpha}_k z}$. Let us denote the orthonormal basis in $\mathbf{H}(B)$ by $\{\varphi_l \,|\, l = 1, \ldots, d\}$ and the orthonormal system by $\phi_{l+kd} = \varphi_l B^k, \quad k \in \mathbb{Z}$. For these systems one can prove the following Marcinkiewicz type theorems:

Theorem 6.1. *Let $f \in \operatorname{span}\{\varphi_l B^k \mid l = 1, \ldots, d, \ |k| < n\}$. Then there exists $C_1 > 0$ such that for $1 \leq p < \infty$ one has*

$$\left(\frac{1}{n}\sum_{k=0}^{n-1} \frac{|f(\zeta_k)|^p}{\sqrt{\beta'(\gamma_k)}}\right)^{\frac{1}{p}} \leq C_1 ||f||_p,$$

and for $1 < p < \infty$ there exists $C_2 > 0$ such that

$$||f||_p \leq C_2 \left(\frac{1}{n}\sum_{k=0}^{n-1} \frac{|f(\zeta_k)|^p}{\sqrt{\beta'(\gamma_k)}}\right)^{\frac{1}{p}}.$$

Proof. For the first part of the assertion let us consider the de La Valée Poussin operators V_n and the fact that $V_{2n} f = f$ and $||V_n|| \leq C$ and by using the Jensen inequality one can obtain: $|f(\zeta_k)| \leq ||V_{2n}|| \, ||f||_p$, i.e.,

$$\frac{|f(\zeta_k)|}{\sqrt{\beta'(\gamma_k)}} \leq ||\gamma'||_\infty ||V_{2n}|| \, ||f||_p,$$

from which follows the assertion.

The proof of the second part uses the fact that for $f \in \mathbb{L}^p$ exists $g \in \mathbb{L}^q$ such that $||g||_q = 1$ and $||f||_p =< f, g >$. Using the Hölder inequality and the result of the previous theorem one can obtain:

$$||f||_p = < f, g > = < f, P_n g > = [f, P_n g]_n^\beta \le$$

$$\le \Big(\frac{1}{n}\sum_{k=0}^{n-1}\frac{|f(\zeta_k)|^p}{\sqrt{\beta'(\gamma_k)}}\Big)^{\frac{1}{p}}\Big(\frac{1}{n}\sum_{k=0}^{n-1}\frac{|g(\zeta_k)|^q}{\sqrt{\beta'(\gamma_k)}}\Big)^{\frac{1}{q}} \le$$

$$\le C||P_n g||_q\Big(\frac{1}{n}\sum_{k=0}^{n-1}\frac{|f(\zeta_k)|^p}{\sqrt{\beta'(\gamma_k)}}\Big)^{\frac{1}{p}} \le C\Big(\frac{1}{n}\sum_{k=0}^{n-1}\frac{|f(\zeta_k)|^p}{\sqrt{\beta'(\gamma_k)}}\Big)^{\frac{1}{p}}. \qquad \Diamond$$

One can observe that for the case when $B(z) = z$ one can reobtain the classical Marcinkiewicz theorems.

By using these results one can prove the following $\mathbb{L}^p$ norm convergence theorem:

Theorem 6.2. *If* $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$ *then for all continuous functions* f *and* $1 < p < \infty$ *one has*

$$\lim_{n\to\infty} ||f - L_n f||_p = 0.$$

Proof. As in the previous proofs, let us consider the de la Valée Poussin operators V_n :

$$||f - L_n f||_p \le ||f - V_n f||_p + ||V_n f - L_n f||_p.$$

Now, by the Th. 6.1. one has

$$||V_n f - L_n f||_p \le C\Big(\frac{1}{n}\sum_{k=0}^{n-1}\frac{|(V_n f - L_n f)(\zeta_k)|^p}{\sqrt{\beta'(\gamma_k)}}\Big)^{\frac{1}{p}} \le$$

$$\le C||f - V_n f||_\infty.$$

By using the fact that $||f - V_n f||_\infty \to 0$ and $||f - V_n f||_p \to 0$ as $n \to \infty$ follows the assertion. $\Diamond$

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