Mathematica Pannonica
9/2 (1998), 267–13

ON A GENERALIZATION OF A FORMULA OF SIERPINSKI, II

Manfred **Kühleitner**

Institut für Mathematik, Universität für Bodenkultur, Gregor Mendel Straße 33, A-1180 Wien, Austria

Received: April 1998

MSC 1991: 11 N 37

Keywords: Divisor problem, asymmetric divisor problem, Dirichlet summatory function, asymptotic expansion.

Abstract: For fixed natural numbers $1 \leq a < b$ we consider $\rho_{a,b}(n)$ defined as the number of pairs $(u,v) \in \mathbb{N} \times \mathbb{Z}, u > |v|$ with $(u-v)^a(u+v)^b = n$. Continuing on part I of this paper we prove an Ω_+- result for the remainder term in the asymptotic formula for the corresponding Dirichlet summatory function.

1. Introduction

As in part I of this paper [11], we define for fixed natural numbers $1 \leq a < b$, the arithmetic function

$$\rho_{a,b}(n) = \#\{(u,v) \in \mathbb{N} \times \mathbb{Z} : u > |v|, (u-v)^a(u+v)^b = n\} \qquad (n \in \mathbb{N}) .$$

To study the average order of this arithmetic function, one is interested in the Dirichlet summatory function

$$T_{a,b}(x) = \sum_{n \leq x} \rho_{a,b}(n) \;, \tag{1.1}$$

where x is a large real variable.

For the special case $a = b = 1$, the question for the asymptotic behaviour of $T_{1,1}(x)$ is closely related to the classical divisor problem of Dirichlet, by the elementary formula, due to Sierpinski [13]

$$\rho_{1,1}(n) = d(n) - 2d\left(\frac{n}{2}\right) + 2d\left(\frac{n}{4}\right) , \tag{1.2}$$

where $d(n)$ denotes the divisor function and $d(\cdot) = 0$ for non-integers. Dirichlet proved that

$$\sum_{n\leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x) , \tag{1.3}$$

where γ denotes the Euler-Mascheroni constant and $\Delta(x) \ll x^{1/2}$. Since then the question of the exact order of the remainder term $\Delta(x)$ has been called the divisor problem of Dirichlet. For an exposition of its history and the definition of the O — and the Ω - symbols, see the textbook of Krätzel [7]. At present, the sharpest upper bound reads

$$\Delta(x) = O\left(x^{23/73}(\log x)^{461/147}\right) , \tag{1.4}$$

due to Huxley [6]. In the opposite direction, the best results to date are

$$\begin{aligned}\Delta(x) = \Omega_+\Big((x \log x)^{1/4}(\log\log x)^{(3+2\log 2)/4}\,\cdot \\ \cdot \exp\left(-c\sqrt{\log\log\log x}\,\right)\Big) \qquad (c > 0) ,\end{aligned} \tag{1.5}$$

and

$$\Delta(x) = \Omega_-\left(x^{1/4} \exp\left(c'(\log\log x)^{1/4}(\log\log\log x)^{-3/4}\right)\right) \qquad (c' > 0) , \tag{1.6}$$

due to Hafner [4], and Corrádi and Kátai [2], respectively.

For the special case $a = b = 1$, (1.2), (1.3) and (1.4) together yield,

$$T_{1,1}(x) = \frac{x}{2} \log x + (2\gamma - 1)\frac{x}{2} + \theta_{1,1}(x) ,$$

with

$$\theta_{1,1}(x) = \Delta(x) - 2\Delta\left(\frac{x}{2}\right) + 2\Delta\left(\frac{x}{4}\right) ,$$

and therefore by (1.4)

$$\theta_{1,1}(x) = O\left(x^{23/73}(\log x)^{461/147}\right) .$$

Concerning lower estimates, the author proved in [9], [10], on the basis of (1.2), Ω - results for $\theta_{1,1}(x)$ which are as sharp as (1.5) resp. (1.6).

In [11], the author showed that for the general case $(a, b) \neq (1, 1)$, there exists a formula quite analogous to (1.2), which is closely related to the asymmetric divisor function

$$d_{a,b}(n) = \sum_{u^a v^b = n} 1 \,, \tag{1.7}$$

and to its corresponding Dirichlet summatory function

$$\sum_{n \le x} d_{a,b}(n) = \zeta\left(\frac{b}{a}\right) x^{1/a} + \zeta\left(\frac{a}{b}\right) x^{1/b} + \Delta_{a,b}(x) \,. \tag{1.8}$$

The corresponding formula in the general case has the form

$$\rho_{a,b}(n) = d_{a,b}(n) - d_{a,b}\left(\frac{n}{2^a}\right) - d_{a,b}\left(\frac{n}{2^b}\right) + 2d_{a,b}\left(\frac{n}{2^{a+b}}\right) \,. \tag{1.9}$$

A thorough account on the history of the asymmetric divisor problem and a survey on results concerning upper estimates for the remainder term $\Delta_{a,b}(x)$ is given in the textbook of Krätzel [7]. The today sharpest lower estimates were established by Hafner [5] and read

$$\begin{aligned} \Delta_{a,b}(x) = \Omega_+ \Big(& x^\alpha (\log x)^{a\alpha} (\log\log x)^{(2\log 2 - 1)a\alpha + 1} \,\cdot \\ & \cdot \exp\left(-c\sqrt{\log\log\log x}\,\right)\Big) \qquad (c > 0) \,, \end{aligned} \tag{1.10}$$

$$\Delta_{a,b}(x) = \Omega_- \left(x^\alpha \exp\left(c'(\log\log x)^{a\alpha} (\log\log\log x)^{a\alpha - 1}\right)\right) \qquad (c' > 0) \,, \tag{1.11}$$

with

$$\alpha = \frac{1}{2(a+b)} \,. \tag{1.12}$$

In [11] the author proved already an Ω_+ - estimate for the error term in the asymptotic expansion of (1.1), quite as sharp as (1.10). The aim of this paper is thus an Ω_-- result for this error term which is as sharp as (1.11).

Here and throughout what follows $c_1, c_2, \ldots$ denote positive constants which depend at most an a, b, which applies to all of the constants implied in the O - and $\ll$ - symbols as well.

Theorem. *For* $1 \le a < b$ *natural numbers, and* α *defined as in* (1.12), *we have*

$$T_{a,b}(x) = \frac{1}{2}\zeta\left(\frac{b}{a}\right)x^{1/a} + \frac{1}{2}\zeta\left(\frac{a}{b}\right)x^{1/b} + \theta_{a,b}(x) ,$$

with

$$\theta_{a,b}(x) = \Omega_-\left(x^\alpha \exp\left(c(\log\log x)^{a\alpha}(\log\log\log x)^{a\alpha-1}\right)\right) ,$$

where c is a positive constant depending on a, b.

2. Notations and Lemmas

For a large real variable x we define P_x as the set of all primes less than or equal to x, and Q_x the set of all a - th powers of square-free integers composed only of primes from P_x. We write N for the cardinality of P_x and $M = 2^{|P_x|}$ for the cardinality of Q_x. We then have

$$N \asymp \frac{x}{\log x} \qquad \text{and} \qquad M \ll \exp\left(c_1 \frac{x}{\log x}\right) ,$$

for some positive constant c_1. The largest integer in Q_x is bounded by e^{2ax}, since for $q \in Q_x$, we have

$$\log q \leq \sum_{p \leq x} a \log p \leq 2ax .$$

Let S_x be the set of numbers defined by

$$S_x = \left\{\mu = \sum_{q \in Q_x} r_q q^{2\alpha} \quad \text{where } r_q \in \{0, \pm 1\} \text{ and at least two } r_q \neq 0\right\} ,$$

and

$$\eta(x) = \inf\{|n^{2\alpha} + 2\mu| \text{ with } n \in \mathbb{N}_0 \text{ and } \mu \in S_x\} .$$

Our first lemma is adopted from Gangadharan [2], and provides an upper bound for $-\log(\eta(x))$, for $x \to \infty$.

Lemma 1. *For $x \to \infty$ we have*

$$\log \frac{1}{\eta(x)} \ll \exp\left(c_2 \frac{x}{\log x}\right) ,$$

for some positive constant c_2.

Proof. Let $h \in \mathbb{N}_0$ and $\mu \in S_x$ such that

$$|h^{2\alpha} + 2\mu| = \eta(x) , \qquad \text{with} \qquad \mu = \sum_{r_q} r_q q^{2\alpha} . \tag{2.1}$$

Then each $1 \neq q \in Q_x$, can be expressed uniquely as a product of distinct primes of the set P_x. Inserting these expressions for the numbers q in (2.1), we get with N as above,

$$\eta(x) = |L(1, h^{2\alpha}, p_1^{2a\alpha}, \dots, p_N^{2a\alpha})| ,$$

where $L(x, y, x_1, \dots, x_N)$ is a polynomial whose degree in each variable is less than or equal one, and whose coefficients are all integers. Let F be the minimal normal extension field of $\mathbb{Q}$ which contains $L(1, h^{2\alpha}, p_1^{2a\alpha}, \dots, p_N^{2a\alpha})$, and $G = \mathrm{Gal}(F/\mathbb{Q})$. Then G contains at most $(a+b)^{N+2}$ elements χ, since the numbers $h^{2\alpha}$, $p_k^{2a\alpha}$ $(1 \le k \le N)$ are all algebraic integers of degree less than or equal $a+b$. It is clear that

$$\left| \prod_{\chi \in G} \chi\left(L(1, h^{2\alpha}, p_1^{2a\alpha}, \dots, p_N^{2a\alpha})\right) \right| \ge 1 , \tag{2.2}$$

since the left hand side of (2.2) is the modulus of the norm of a nonzero algebraic integer. (Note that $q_1^{2\alpha}, \dots, q_M^{2\alpha}$ are linearly independent over $\mathbb{Q}$, see e.g. [1].) Furthermore, for every $\chi \in G$,

$$\chi\left(L(1, h^{2\alpha}, p_1^{2a\alpha}, \dots, p_N^{2a\alpha})\right) \le \eta(x) + 4 \max_{1 \le k \le N} p_k^{2a\alpha} . \tag{2.3}$$

From (2.2), (2.3) we obtain

$$\frac{1}{\eta(x)} \le \prod_{\substack{\chi \in G \\ \chi \ne id}} \chi\left(L(1, h^{2\alpha}, p_1^{2a\alpha}, \dots, p_N^{2a\alpha})\right) \le (1 + 4x^{2a\alpha})^{(a+b)^{N+2}} ,$$

which establishes Lemma 1. ◊

Lemma 2. *For $(u, v) \in \mathbb{N}^2$ let*

$$\tau_{a,b}(n) = \sum_{u^a v^b = n} u^{a-1} v^{b-1} . \tag{2.4}$$

There exists a positive constant c_3 such that

$$\sum_{q \in Q_x} \frac{\tau_{a,b}(q)}{q^{1-\alpha}} \gg \exp\left(c_3 \frac{x^{a\alpha}}{\log x}\right) .$$

Proof. By the definition of Q_x, we have

$$\sum_{q \in Q_x} \frac{\tau_{a,b}(q)}{q^{1-\alpha}} \ge \prod_{p \le x} (1 + p^{-1+a\alpha}) = \exp\left(\sum_{p \le x} \log(1 + p^{-1+a\alpha})\right) \ge$$

$$\ge \exp\left(\sum_{p \le x} p^{-1+a\alpha} + O(1)\right) \gg \exp\left(c_3 \frac{x^{a\alpha}}{\log x}\right) . \qquad \Diamond$$

Lemma 3. *For $\tau_{a,b}(n)$ defined as in (2.4), we have*

$$\tau_{a,b}(2^{a+b}n) = 2^{a-1}\tau_{a,b}(2^b n) + 2^{b-1}\tau_{a,b}(2^a n) - 2^{a+b-2}\tau_{a,b}(n) .$$

Proof. Write $n = 2^{a+b+r}u$, with u odd. Then

$$\begin{aligned}\tau_{a,b}(2^{a+b+r}) &= \sum_{u^a v^b = 2^{a+b+r}} u^{a-1}v^{b-1} = \\ &= \Big\{ \sum_{\substack{u^a v^b = 2^{a+b+r} \\ 2|u}} + \sum_{\substack{u^a v^b = 2^{a+b+r} \\ 2|v}} - \sum_{\substack{u^a v^b = 2^{a+b+r} \\ 2|u, 2|v}} \Big\} u^{a-1}v^{b-1} = \\ &= \Big\{ 2^{a-1} \sum_{u^a v^b = 2^{b+r}} + 2^{b-1} \sum_{u^a v^b = 2^{a+r}} - 2^{a+b-2} \sum_{u^a v^b = 2^r} \Big\} u^{a-1}v^{b-1} = \\ &= 2^{a-1}\tau_{a,b}(2^{b+r}) + 2^{b-1}\tau_{a,b}(2^{a+r}) - 2^{a+b-2}\tau_{a,b}(2^r)\end{aligned}$$

The proof now follows from the multiplicativity of $\tau_{a,b}(\cdot)$. ◊

As in Gangadharan [2], define for real z,

$$V(z) = 2\left(\cos\left(\frac{z}{2}\right)\right)^2 = 1 + \frac{e^{iz} + e^{-iz}}{2} ,$$

and

$$T_x(u) = \prod_{q \in Q_x} V\left(uq^{2\alpha} - \frac{5\pi}{4}\right) .$$

Lemma 4. *We have*

(1) $0 \le T_x(u) \le 2^M$*, for all* u*;*
(2) $T_x'(u) \ll M\,2^M\,e^{2ax}$*, for all* u*;*
(3) $T_x(u) = T_0 + T_{1,x} + T_{2,x} + T_{3,x}$ *where,*

$$T_0 = 1, \quad T_{1,x} = \frac{e^{5\pi i/4}}{2} \sum_{q \in Q_x} e^{-iuq^{2\alpha}}, \quad T_{3,x} = \sum_{\mu \in S_x} h_\mu e^{iu\mu} ,$$

$T_{2,x}$ is the complex conjugate of $T_{1,x}$ and $|h_\mu| \le 1/4$.

Proof. The proof of Lemma 3 is straightforward by the definition of $V(z)$ and $T_x(u)$. ◊

3. Proof of the Theorem

We start from formulas (47), (48) of Krätzel [8], with a slight change of notation: For $x \ge 0$, we have

(3.1)
$$\Delta_1(x) \stackrel{\text{def}}{=} \int_1^x \left(\Delta_{a,b}(t) - \frac{1}{4} \right) dt =$$
$$= c_4 x^{1-\alpha} \sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^{1+\alpha}} \sin\left(c_5 (nx)^{2\alpha} - \frac{\pi}{4}\right) + O(x^{1-2\alpha}) ,$$

where the sum converges absolutely and uniformly on every compact set, and c_4, c_5 are explicit computable positive constants, e.g.

$$c_4 = \frac{ab(a^b b^a)^{-2\alpha}}{2\pi^2 \sqrt{a+b}} , \qquad c_5 = 2\pi(a+b)(a^a b^b)^{-2\alpha} .$$

For the error term in (3.1) see Nowak [12], formula (2.18). Let

$$E(t) = c_6 \left(\theta_{a,b} \left((c_7 t)^{a+b} \right) - \frac{1}{4} \right) ,$$

with $c_6 = \frac{a+b}{c_4 \sqrt{c_5}}$ and $c_7 = c_5^{-1}$. From (1.9), (3.1) and the substitution $T = c_5 x^{2\alpha}$, we get

(3.2)
$$E_1(T) \stackrel{\text{def}}{=} \int_0^T E(t) t^{a+b-1} dt =$$
$$= T^{a+b-1/2} \sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^{1+\alpha}} \left(s_0(n,T) - s_a(n,T) - s_b(n,T) + 2 s_{a+b}(n,T) \right) +$$
$$+ O(T^{a+b-1})$$

with

$$s_e(n,T) := 2^{e\alpha} \sin\left(T(n 2^{-e})^{2\alpha} - \frac{\pi}{4} \right) .$$

For $c_8 = \max\{c_2, 2c_1\}$ we define

$$P(x) = \exp\left(c_8 \frac{x}{\log x} \right) \qquad \text{and} \qquad \sigma_x = \exp\left(-2P(x)\right) .$$

Therefore $M = o(P(x)$ and $-\log \eta(x) = o(P(x))$, too. Next define for fixed x,

$$\gamma_x = \sup_{u>0} \frac{-c_6\, \theta_{a,b} \left((c_7 u)^{a+b} \right)}{u^{1/2 + 1/P(x)}} .$$

We may assume that $\gamma_x < \infty$, otherwise more than Theorem would be true. With $A = c_6/4$, we have

$$\gamma_x u^{1/2+1/P(x)} + A + E(u) \geq 0 \ , \tag{3.3}$$

for all u. Let

$$J_x = \sigma_x^{a+b+1/2} \int_0^\infty \left(\gamma_x u^{1/2+1/P(x)} + A + E(u)\right) u^{a+b-1} \, \mathrm{e}^{-\sigma_x u} \, T_x(u) \, du.$$

The next lemma provides an asymptotic expansion for J_x.

Lemma 5. *For $x \to \infty$ and α as in* (1.12),

$$J_x = e^2 \, \Gamma(a+b+1/2)\gamma_x - \frac{1}{4}\Gamma(a+b+1/2) \sum_{q \in Q_x} \frac{\tau_{a,b}(q)}{q^{1-\alpha}} + o(\gamma_x) + o(1) \ .$$

Proof. Throughout this proof, we write $\kappa = a+b+1/2$, for short. Do deal with the first two terms of J_x, we observe that, for $r = a+b-1$ or $r = a+b-\frac{1}{2}+\frac{1}{P(x)}$,

$$\sigma_x^\kappa \int_0^\infty u^r \, \mathrm{e}^{-\sigma_x u} \, T_x(u) \, du =$$

$$= \Gamma(1+r) \, \sigma_x^{\kappa-1-r} + \sum_{i=1,2,3} \sigma_x^\kappa \int_0^\infty u^r \, \mathrm{e}^{-\sigma_x u} \, T_{i,x}(u) \, du \ ,$$

where $1 \leq r \leq a+b-1/2+1/P(x)$. The part of $T_{1,x}$ contributes exactly,

$$\frac{\mathrm{e}^{5\pi i/4}}{2} \sigma_x^\kappa \Gamma(1+r) \sum_{q \in Q_x} \frac{1}{(\sigma_x + iq^{2\alpha})^{1+r}} \ll \sigma_x^\kappa \sum_{q \in Q_x} q^{-2\alpha(1+r)} \ll$$

$$\ll \sigma_x^\kappa \sum_{q \in Q_x} 1 \ll \sigma_x^\kappa M = o(1) \ .$$

The contribution of $T_{2,x} = \overline{T_{1,x}}$ is obviously no more than this. Finally $T_{3,x}$ contributes

$$\sigma_x^\kappa \sum_{\mu \in S_x} \frac{h_\mu}{(\sigma_x + i\mu)^{1+r}} \ll \sigma_x^\kappa 3^M \eta(x)^{-(1+r)} \ll$$

$$\ll \exp\left(-2\kappa P(x) + M \ln 3 + (1+r)(-\log \eta(x))\right) \ll$$

$$\ll \exp\left(-2\kappa P(x) + o(P(x))\right) = o(1) \ .$$

Next we deal with the contribution of $E(u)$ to J_x. Our first step is to integrate by parts to introduce $E_1(u)$ in the integral. Thus,

$$I \overset{\text{def}}{=} \int_0^\infty E(u) u^{a+b-1} \, \mathrm{e}^{-\sigma_x u} \, T_x(u) \, du =$$

$$= - \int_0^\infty E_1(u) \frac{d}{du} \big(\mathrm{e}^{-\sigma_x u} \, T_x(u) \big) \, du \ ,$$

since $E_1(u) \ll u^{a+b-1/2}$ for large u and $E_1(0) = 0$. Inserting the series representation (3.2) for $E_1(u)$ and integrating term by term, noting that the series converges absolutely for every u and uniformly on compact sets, we get

$$I = - \sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^{1+\alpha}} \mathrm{Im}(\mathrm{e}^{-\pi i/4} \, I_n) + O\Big(\int_0^\infty \big| \frac{d}{du} \big(\mathrm{e}^{-\sigma_x u} \, T_x(u) \big) \big| \, du \Big) +$$

$$+ O\Big(\int_0^\infty u^{a+b-3/2} \, \mathrm{e}^{-\sigma_x u} \, |T_x(u)| \, du \Big),$$

since

$$u^{a+b-1/2} \frac{d}{du} \big(\mathrm{e}^{-\sigma_x u} \, T_x(u) \big) = \frac{d}{du} \big(u^{a+b-1/2} \, \mathrm{e}^{-\sigma_x u} \, T_x(u) \big) +$$

$$+ O\big(u^{a+b-3/2} \, \mathrm{e}^{-\sigma_x u} \, T_x(u) \big) \ ,$$

and

$$I_n \overset{\text{def}}{=}$$

$$= \int_0^\infty \big(e(n;0) - e(n;a) - e(n;b) + 2e(n;a+b) \big) \frac{d}{du} \big(u^{a+b-1/2} \, \mathrm{e}^{-\sigma_x u} \, T_x(u) \big) du,$$

with

(3.4) $$e(n;r) := 2^{r\alpha} \, \mathrm{e}^{iu(n/2^r)^{2\alpha}} \ .$$

Estimating the contributions of the error terms, we see that

$$\int_0^\infty \Big| \frac{d}{du} \big(\mathrm{e}^{-\sigma_x u} \, T_x(u) \big) \Big| \, du \le \int_0^\infty |T_x(u)' - \sigma_x \, T_x(u)| \, \mathrm{e}^{-\sigma_x u} \, du \le$$

$$\le 4^M \sigma_x^{-1} + 2^M \ll$$

$$\ll \exp\big(M \ln 4 + 2P(x) \big) + o(1) = o(\sigma_x^{-\kappa}) \ ,$$

since $\kappa > 2$, and

$$\int_0^\infty u^{a+b-3/2}\, e^{-\sigma_x u}\, |T_x(u)|\, du \ll$$

$$\ll 2^M \int_0^\infty u^{a+b-3/2}\, e^{-\sigma_x u}\, du \ll 2^M \sigma_x^{-(a+b-1/2)} \ll$$

$$\ll \exp\big(2(a+b-1/2)P(x) + o(P(x))\big) = o(\sigma_x^{-\kappa}) .$$

We integrate I_n by parts once more and expand $T_x(u)$ as in (3) of Lemma 4, to get with

$$e_1(n;r) := \frac{n}{2^r}^{2\alpha} e(n,r) ,$$

$e(n;r)$ as in (3.4),

$$I_n = -i(I_0(n) + I_1(n) + I_2(n) + I_3(n)) ,$$

with

$$I_k(n) =$$

$$= \int_0^\infty \big(e_1(n;0) - e_1(n;a) - e_1(n;b) + e_1(n;a+b)\big) u^{\kappa-1}\, e^{-\sigma_x u}\, T_{k,x}(u)\, du$$

for $0 \le k \le 3$. We shall show that the main term of I_n comes from $I_1(n)$. In fact, the contribution of $I_0(n)$ is

$$\ll n^{2\alpha}\, |\sigma_x - in^{2\alpha}|^{-\kappa} \ll n^{-1+\alpha} ,$$

that of $I_2(n)$ is

$$\ll n^{2\alpha} \sum_{q \in Q_x} |\sigma_x - i(n^{2\alpha} + q^{2\alpha})|^{-\kappa} \ll M n^{-1+\alpha} .$$

The contribution of $I_3(n)$ is bounded by

$$I_3(n) \ll n^{2\alpha} \sum_{\mu \in S_x} |\sigma_x - i(n^{2\alpha} - \mu)|^{-\kappa} \ll$$

$$\ll \begin{cases} n^{2\alpha}\, 3^M (\eta(x))^{-\kappa} , & \text{if } n \le 2 \max\{|\mu| : \mu \in S_x\} \\ n^{-1+\alpha}\, 3^M , & \text{else.} \end{cases}$$

This $\max\{|\mu| : \mu \in S_x\}$ is bounded by $M\, e^{cx}$ for some positive constant c. Hence the total contribution to I is bounded by

$$\ll \sum_{n\le 2M\,\mathrm{e}^{cx}} \frac{\tau_{a,b}(n)}{n^{1-\alpha}}\, 3^M \exp\left(-\kappa \log \eta(x)\right) + O\Big(3^M \sum_{n>2M\,\mathrm{e}^{cx}} \frac{\tau_{a,b}(n)}{n^2}\Big) \ll$$
$$\ll 3^M \sigma_x^{-\kappa \log \eta(x)} (M\,\mathrm{e}^{cx})^{\alpha+\epsilon} = o(\sigma_x^{-\kappa}) \;.$$

Therefore,

$I =$

$$-\frac{1}{2}\sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^{1-\alpha}} \sum_{q\in Q_x} \int_0^{\infty} \left(e_q(n;0) - e_q(n;a) - e_q(n;b) + 2e_q(n;a+b)\right) u^{\kappa-1} \mathrm{e}^{-\sigma_x u}\, du +$$
$$+\, o(\sigma_x^{-\kappa}) =$$
$$-\frac{1}{2}\sum_{q\in Q_x} \frac{1}{q^{1-\alpha}} \Big(\tau_{a,b}(q) - 2^{-a}\tau_{a,b}(2^a q) - 2^{-b}\tau_{a,b}(2^b q) + 2^{-a-b+1}\tau_{a,b}(2^{a+b} q)\Big) \cdot$$
$$\cdot \int_0^{\infty} u^{a+b-1/2}\, \mathrm{e}^{-\sigma_x u}\, du + O\Big(\sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^{1-\alpha}} \sum_{\substack{q\in Q_x\\ n\ne q}} \Big|\int_0^{\infty} \mathrm{e}^{iu(n^{2\alpha}-q^{2\alpha})}\, u^{\kappa-1}\, \mathrm{e}^{-\sigma_x u}\Big|\, du\Big)$$

with

$$e_q(n;r) := 2^{r\alpha}\left(\frac{n}{2^r}\right)^{2\alpha} \mathrm{e}^{iu((n/2^r)^{2\alpha}-q^{2\alpha})} \;.$$

For this last error term we get a bound exactly as above for $I_3(n)$ with M replacing the factor 3^M.

Combining all contributions we get,

$$I = -\frac{\Gamma(\kappa)}{2} \sigma_x^{-\kappa} \sum_{q\in Q_x} q^{-1+\alpha} \big(\tau_{a,b}(q) - 2^{-a}\tau_{a,b}(2^a q) - 2^{-b}\tau_{a,b}(2^b q) +$$
$$+\, 2^{-a-b+1}\tau_{a,b}(2^{a+b} q)\big) + o(\sigma_x^{-\kappa}) =$$
$$= -\frac{1}{4}\Gamma(\kappa)\sigma_x^{-\kappa} \sum_{q\in Q_x} \tau_{a,b}(q) q^{-1+\alpha} + o(\sigma_x^{-\kappa}) \;,$$

the last assertion by Lemma 3. This completes the proof of Lemma 5. ◇

Since $\sigma_x > 0$ and $J_x > 0$ by (3.3), we have

$$\exp\left(c\frac{x^{a\alpha}}{\log x}\right) \ll \sum_{q\in Q_x} \tau_{a,b}(q) q^{-1+\alpha} \ll \gamma_x \;,$$

by Lemma 2 and Lemma 5. Thus by the definition of γ_x there is a sequence u_x which tends to infinity with x, such that

$$-\theta_{a,b}(u_x^2) \gg u_x^{1/2} \exp\left(\frac{\log u_x}{P(x)} + c\frac{x^{a\alpha}}{\log x}\right) \;,$$

since $\theta_{a,b}(u)$ is bounded for bounded u, which follows for small u from

$$\theta_{a,b}(u) = -\frac{1}{2}\zeta\Big(\frac{b}{a}\Big)x^{1/a} - \frac{1}{2}\zeta\Big(\frac{a}{b}\Big)x^{1/b}$$

and is obvious for the other values of u.

Consider first the values of u_x for which

$$\frac{\log u_x}{P(x)} \le c\frac{x^{a\alpha}}{\log x} . \tag{3.5}$$

Taking logarithms on both sides, we have

$$\log\log u_x \ll \frac{x}{\log x} .$$

Since $y^{a\alpha}(\log y)^{-1+a\alpha}$ is an increasing function of y for sufficiently large y, we have from (3.5)

$$\frac{(\log\log u_x)^{a\alpha}}{(\log\log\log u_x)^{1-a\alpha}} \ll \frac{x^{a\alpha}}{\log x} ,$$

from which the desired estimate follows.

Consider now thoose values of x for which

$$c\frac{x^{a\alpha}}{\log x} \le \frac{\log u_x}{P(x)} . \tag{3.6}$$

We may assume that

$$\frac{(\log\log u_x)^{a\alpha}}{(\log\log\log u_x)^{1-a\alpha}} \gg \frac{\log u_x}{P(x)} ,$$

otherwise the estimate holds obviously. Taking logarithms on both sides gives

$$\log\log u_x \ll \frac{x}{\log x} ,$$

from which the estimate follows as above. This proves the Theorem. ◊

References

[1] BESICOVITCH, A. S.: On the linear independence of fractional powers of integers, *J. London Math. Soc.* **15** (1940) 3–6.

[2] CORRÁDI I. K. and KÁTAI, F.: Egy megjegyzés K.S. Gangadharan "Two classical lattice point problems" című dolgozatához, *MTA III Osztály Közleményei* **17** (1967), 89–97.

[3] GANGADHARAN, K. S.: Two classical lattice point problems, *Proc. Cambridge Phil. Soc.* **57** (1961), 699–721.

[4] HAFNER, J. L.: New omega theorems for two classical lattice point problems, *Invent. Math.* **63** (1981), 181–186.

[5] HAFNER, J. L.: New omega theorems in a weighted divisor problem, *J. Number Theory* **28**, (1988), 240–257.

[6] HUXLEY, M. N.: Exponential sums and lattice points II, *Proc. London Math. Soc.* **66**/3 (1993), 279–301.

[7] KRÄTZEL, E.: "Lattice Points", Berlin, 1988.

[8] KRÄTZEL, E.: Ein Teilerproblem, *J.Reine Angew. Math.* **235** (1969), 150–174.

[9] KÜHLEITNER, M.: An omega theorem on differences of two squares, *Acta Math. Univ. Comenianae* **61**/1 (1992), 117–123.

[10] KÜHLEITNER M.: An omega theorem on differences of two squares II, *to appear.*

[11] KÜHLEITNER, M.: On a generalization of a formula of Sierpinski, *Mathematica Pannonica* **9**/1 (1998), 91–101.

[12] NOWAK, W. G.: A two-sided omega-theorem for an asymmetric divisor problem, *Manuscripta Math.* **69** (1990), 153–172.

[13] SIERPINSKI, W.: Über die Darstellungen ganzer Zahlen als Differenz von zwei Quadraten, *Wiadom. Mat.* **11** (1907), 89–110.