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A SPECIAL SOLUTION OF THE INHOMOGENEOUS SCHRÖDER EQUATION

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Abstract: A unique solution $\varphi : [0, a) \to \mathbb{R}$ of equation (p) is found in the class of continuous functions in $[0, a)$ and such that $\varphi(x) \sim z(x0$ as $x \to 0+$. The result is applied to determine a unique solution to inequality (p, q) of second order.

The equation in question reads

$$\varphi(f(x)) = p\,\varphi(x) + z(x) \tag{p}$$

where the given function f — a selfmapping of an interval — is subjected to conditions (H) specified below, p is a real number and φ is the unknown function.

In the theory of iterative functional equations the most frequent is the case where solutions form a large class of functions depending on an arbitrary function. Thus conditions are wanted that ensure the

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existence of a unique (or a one-parameter family, say) solution to the equation considered, satisfying these conditions.

In this paper we are interested in continuous solutions of (p) that are asymptotically comparable with the function z at the origin (which is chosen to be the only fixed point of the function f in $\operatorname{Dom} f$). We prove that such a solution is unique.

The paper is concluded with comments on special solutions ψ of the functional inequality of second order ($f^2 := f \circ f$):

$$\psi(f^2(x)) \leq (p+q)\psi(f(x)) - pq\psi(x) \tag{p,q}$$

to which the result concerning equation (p) can be applied, cf. [2]. In this way we supply a new proof of Th. 12.7.4 in [4], the original proof of which requires a correction, as it was pointed out by Maria Stopa.

Main result

We denote

$$I := [0,a) \quad \text{or} \quad I := [0,a], \quad a > 0, \quad \text{and} \quad I^* := I \setminus \{0\}.$$

The function f is subjected to the following hypotheses:

(H) $f : I \to I$ is continuous and strictly inctreasing in I,
$0 < f(x) < x$ for $x \in I^*$.

By f^n we denote the n-th functional iterate of the function f, i.e.,

$$f^0 = \mathrm{id}_I, \quad f^{n+1} = f \circ f^n, \quad n \in \mathbb{N} \cup \{0\}.$$

We are looking for solutions $\varphi : I \to \mathbb{R}$ of equation (p) in I continuous on I and satisfying the limit condition

$$L_\varphi := \lim_{x \to 0+} [\varphi(x)/z(x)] < \infty. \tag{L}$$

We shall prove the following

Theorem. *Assume* (H) *and let* $z : I \to (0, +\infty)$ *be a continuous function on* I *satisfying the condition*

$$0 < c := \lim_{x \to 0+} z(f(x))/z(x) < p. \tag{c}$$

Then the function $\varphi_0 : I \to \mathbb{R}$ *given by the formula*

$$\varphi_0(x) = -\sum_{n=0}^{\infty} p^{-n-1} z(f^n(x)), \quad x \in I, \tag{S}$$

is the unique continuous solution of (p) *in* I *satisfying the limit condition* (L). *Moreover,*

$$L_{\varphi_0} = (c-p)^{-1}.$$

Proof. Assume that $\varphi : I \to \mathbb{R}$ is a continuous solution to (p) satisfying (L). We first calculate L_φ, using (p):

$$\frac{\varphi(f(x))}{z(x)} = p\,\frac{\varphi(x)}{z(x)} + 1.$$

On passing to the limits as $x \to 0+$ we get by (c) and (L):

$$c\,L_\varphi = p\,L_\varphi + 1,$$

as (cf. (H))

$$\lim_{x\to 0+}\left[\frac{\varphi(f(x))}{z(f(x))}\,\frac{z(f(x))}{z(x)}\right] = \lim_{x - 0+}\left[\frac{\varphi(f(x))}{z(x)}\right].$$

Thus $L_\varphi = (c-p)^{-1}$.

Now, consider the function $\alpha : I \to \mathbb{R}$ defined by

(A) $$\alpha(x) = \begin{cases} \varphi(x)/z(x), & x \in I^* \\ L_\varphi, & x = 0. \end{cases}$$

This is a function continuous on I. Moreover, φ is a continuous solution of (p) satisfying (L) with $L_\varphi = (c-p)^{-1}$ iff α is a continuous solution to the equation

(α) $$\alpha(f(x)) = p\,\frac{z(x)}{z(f(x))}\,\alpha(x) + \frac{z(x)}{z(f(x))}, \quad x \in I,$$

where the value at zero of the function $z/z\circ f$ is defined as c^{-1}, cf. (c).

We shall prove that equation (α) has the unique solution continuous on the whole I (crucial is the continuity at zero). To this end we need the following fact taken from [3], cf. also Ths. 3.1.10 and 3.1.9 from [4], pp. 103–104, and reformulated accordingly.

Lemma. *If hypotheses* (H) *are fulfilled,* $g : I \to \mathbb{R}$ *is continuous,* $g(x) \neq 0$ *in* I^*, $h : I \to \mathbb{R}$ *is continuous on* I *and*

(*) $$|g(0)| > 1$$

then the equation

$$\alpha(f(x)) = g(x)\alpha(x) + h(x)$$

has the unique solution $\alpha : I \to \mathbb{R}$ *continuous on* I *which is given by the formula*

(**) $$\alpha(x) = d - \sum_{n=0}^{\infty} \frac{h(f^n(x); d)}{G_{n+1}(x)}, \quad x \in I,$$

where

$$d = h(0)/(1 - g(0)),$$
$$h(x; d) = h(x) + d(g(x) - 1), \quad x \in I,$$
$$G_1(x) = g(x), \quad G_{n+1}(x) = G_n(x)g(f^n(x)), \quad n \in \mathbb{N}, \quad x \in I.$$

In our case
$$h(x) = z(x)/z(f(x)), \quad x \in I^*, h(0) = c^{-1}; g(x) = p\,h(x), \quad x \in I,$$
cf. (c). It is seen that these functions satisfy the general hypotheses of the Lemma. Condition (*) follows from (c):
$$0 < g(0) = p\,h(0) = p\,c^{-1} > 1.$$
Thus there exists the unique continuous solution α of (α) in I.

We claim that $\varphi_0 = z\alpha$ where φ_0 is given by (S).

First of all, the series in (S) actually converges, since it has a convergent (numerical) majorant. For, assumption (c) yields, with $a, b \in \in (0, c)$, the existence of a $\delta > 0$ such that

(b) $$0 < \frac{z(f(x))}{z(x)} \leq b \quad \text{for} \quad x \in (0, \delta).$$

By (H) the sequence $(f^n(x))_{n\in\mathbb{N}}$ is strictly decreasing (for each $x \in I^*$) and it converges to zero (the fixed point of f). If $a \in I$ we may find such an $n_b \in \mathbb{N}$ that $f^n(a) \in (0, \delta)$ for $n \geq n_b$ (if $a \notin I$ we take any $a' \in I^*$ and argue in the same way). Consequently $f^n(x) \in (0, \delta)$ for every $x \in I$ and $n \geq n_b$. Now, in virtue of (b) we obtain, whenever $n \geq n_b$ and $x \in I$,
$$0 < z(f^n(x)) \leq b\,z(f^{n-1}(x)) \leq \ldots \leq b^{n-n_b+1}z(f^{n_b}(x)) \leq M\,b^{n+1}$$
(since the function $z \circ f^{n_b}$ is continuous on I), and also
$$0 < p^{-n-1}z(f^n(x)) \leq M\left(\frac{b}{p}\right)^{n+1} \quad \text{for} \quad x \in I, \quad n \geq n_b,$$
where $0 < b/p < c/p < 1$, by (c). Therefore the series from (S) actually converges in I (in fact, uniformly or almost uniformly according to whether a belongs to I or not).

Now we examine formula (**), checking step by step:

(d) $$d = c^{-1}(1 - p\,c^{-1})^{-1} = (c - p)^{-1},$$

$$h(x; d) = h(x) + d(ph(x) - 1) = d(ch(x) - 1), \quad x \in I,$$

$$G_n(x) = p^n z(x)/z(f^n(x)), \quad n \in \mathbb{N}, \quad x \in I,$$

$$\sum_{n=0}^{\infty} \frac{h(f^n(x);d)}{G_{n+1}(x)} = d \sum_{n=0}^{\infty} \left[c \frac{z(f^n(x))}{z(f^{n+1}(x))} - 1 \right] p^{-n-1} \frac{z(f^{n+1}(x))}{z(x)} =$$

$$= \frac{d}{z(x)} \left(c \sum_{n=0}^{\infty} p^{-n-1} z(f^n(x)) - p \sum_{n=0}^{\infty} p^{-n-2} z(f^{n+1}(x)) \right)$$

(since both series are convergent). Formula (**) then yields

$$\alpha(x) = d - \sum_{n=0}^{\infty} \frac{h(f^n(x);d)}{G_{n+1}(x)} =$$

$$= d - \frac{d}{z(x)} \left(c \sum_{n=0}^{\infty} p^{-n-1} z(f^n(x)) - p \sum_{m=0}^{\infty} p^{-m-1} z(f^m(x)) + z(x) \right)$$

$$= d(c-p) \frac{\varphi_0(x)}{z(x)}, \quad x \in I^*,$$

cf. (S). Thus our claim follows from (d). ◊

A functional inequality of second order

The functional inequality

(p,q) $$\psi(f^2(x)) \leq (p+q)\psi(f(x)) - pq\psi(x)$$

is equivalent to the system consisting of equation (p) and the inequality

(q) $$z(f(x)) \leq q\, z(x)$$

in the following sense: if $\psi : I \to \mathbb{R}$ is a continuous solution of inequality (p,q) then, on putting

(z) $$z(x) = \psi(f(x)) - p\psi(x), \quad x \in I,$$

we obtain a continuous solution of (q):

$$z(f(x)) = \psi(f^2(x)) - p\psi(f(x)) \leq q\psi(f(x)) - pq\psi(x) =$$

$$= q[\psi(f(x)) - p\psi(x)] = qz(x);$$

and vice versa: if $z : I \to \mathbb{R}$ satisfies (q) and $\varphi : I \to \mathbb{R}$ is a solution to equation (p) (both function being continuous in I) then this φ satisfies (p,q):

$$\varphi(f^2(x)) - p\varphi(f(x)) = z(f(x)) \leq q\, z(x) = q(\varphi(f(x)) - p\varphi(x)).$$

For inequalities of first order $(\beta(f(x)) \leq g(x)\beta(x))$ the notion of a regular solution has been introduced by D. Brydak in [1]. When adapted to (q) the definition reads (cf. also [4], p. 473).

Definition. A continuous solution $z : I \to \mathbb{R}$ of (q) is said to be *regular* iff there exists a continuous solution $\sigma : I \to \mathbb{R}$ of the equation

$$\sigma(f(x)) = q\sigma(x)$$

such that $\sigma \leq z$ and the function $\sigma_z : I \to \mathbb{R}$ defined by

$$\sigma_z(x) = \lim_{n\to\infty}[q^{-n}z(f^n(x))], \quad x \in I^*, \quad \sigma_z(0) = 0,$$

is continuous on I.

Our Th. yields the following result for special solutions of inequality (p,q)

Proposition. *Let hypotheses* (H) *be satisfied and let* $0 < q < p$. *If* z *is a regular solution of inequality* (q) *and* $\alpha_z(x) > 0$ *in* I^* *then there exists the unique continuous solution* $\varphi_0 : I \to \mathbb{R}$ *of inequality* (p,q) *fulfill* (L). *This solution satisfies* (p), *is given by* (S), *and* $L_{\varphi_0} = (q-p)^{-1}$.

Proof. The assumptions of the Prop. imply (cf. Th. 12.4.7 in [4], p. 487) that

$$c = \lim_{x\to 0} z(f(x))/z(x) = q.$$

Thus the statement actually follows from our Th. $\Diamond$

Remark. The Prop. is strictly related to Th. 12.7.4 from [4], p. 497. However, a part of the proof of that Theorem (p. 498, lines 4–8. from above) is incorrect because of using the auxiliary equation (cf. line 4.)

$$\psi(f(x)) = [p\,z(x)/z(f(x))]\psi(x) + 1$$

instead of our (α). In this paper we gave an independent proof of Th. 12.7.4 from [4].

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