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ON THE VOLTERRA INTEGRAL EQUATION AND THE HENSTOCK-KURZWEIL INTEGRAL

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Abstract: In this paper we prove that the set of all continuous solutions of the nonlinear Volterra integral equation provided with the Henstock–Kurzweil integral is an R_δ. In particular, it is nonempty, compact and connected in the space of all continuous functions with the topology of uniform convergence.

The Henstock–Kurzweil (shortly: (H-K)) integral (see [4], [6]) which is equivalent to the Denjoy-Perron one, is based on the modification of the Riemann original definition. Since the Henstock–Kurzweil integral allows to integrate the Newton, Riemann and Lebesgue integrable functions, so it is a convenient tool to investigate so called generalized solutions of differential equation (see [2], [3], [5], [6], [7]).

The purpose of this paper is to investigate continuous solutions of the following nonlinear Volterra integral equation

$$x(t) = q(t) + \int_0^t f(t, s, x(s))ds, \quad t \in I, \tag{1}$$

where $I = [0, a]$, $a > 0$ and the sign "$\int$" stands for the (H-K)-integral. In what follows we obtain the topological characterization (in particular

— the existence) of the continuous solutions set of (1). Our result generalizes Th. 1 and Th. 4 ([2]).

Assume that $T = \{(t,s) : 0 \le s \le t \le a\}$ and

1^0 $q : I \to (\alpha, \beta)$ is a continuous function;

2^0 $f : T \times [\alpha,\beta] \to \mathbb{R}$ is a function such that

(i) $s \to f(t,s,x)$ is a Lebesgue measurable function for every $x \in [\alpha,\beta]$ and $t \in I$,

(ii) $x \to f(t,s,x)$ is a continuous function for every $t \in I$ and a.e. $s \in [0,t]$;

3^0 there exist functions $m : T \to \mathbb{R}$ and $M : T \to \mathbb{R}$ such that for every $t \in I$ the functions $m(t,\cdot)$ and $M(t,\cdot)$ are integrable in the Henstock–Kurzweil sense on $[0,t]$ and
$m(t,s) \le f(t,s,x) \le M(t,s)$ for $(t,s,x) \in T \times [\alpha,\beta]$;

4^0 there exist functions $r(\tau,t,s)$, $R(\tau,t,s)$ $(0 \le s \le t \le \tau \le a)$ such that the functions $r(\tau,t,\cdot)$, $R(\tau,t,\cdot)$ are (H-K)-integrable and

$$r(\tau,t,s) \le f(\tau,s,x) - f(t,s,x) \le R(\tau,t,s);$$

5^0 $\lim\limits_{\tau - t \to 0^+} [\int_0^t r(\tau,t,s)ds + \int_t^\tau m(\tau,s)ds] = 0$ and

$\lim\limits_{\tau - t \to 0^+} [\int_0^t R(\tau,t,s)ds + \int_t^\tau M(\tau,s)ds] = 0$ for fixed t or τ.

Denote by $C(I,[\alpha,\beta])$ the topological space of all continuous functions $I \to [\alpha,\beta]$ with the topology of uniform convergence. Now we prove the following

Theorem. *Under the above assumptions there exists an interval $J \subset I$ such that for $B = C(J,[\alpha,\beta])$ and $G(x)(t) = q(t) + \int_0^t f(t,s,x(s))ds$*

(i) *$G(B)$ is relatively compact;*

(ii) *G is continuous.*

Hence G satisfies the assumptions of the Vidossich Theorem ([8], Cor. 1.2.) and therefore one has the following

Corollary. *The set S of all continuous solutions of (1) is an R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

Proof of the Theorem Let $\alpha_1 = \min\limits_{t \in I} q(t)$, $\beta_1 = \max\limits_{t \in I} q(t)$. Obviously $\alpha_1 > \alpha$ and $\beta_1 < \beta$. By 5^0, it is clear that we can choose a number $0 < d \le a$ such that

$$\int_0^t M(t,s)ds \le \beta - \beta_1 \quad \text{and} \quad \int_0^t m(t,s)ds \ge \alpha - \alpha_1$$

for every $t \in [0,d]$. Put $J = [0,d]$. Define

$$F(x)(t) = \int_0^t f(t,s,x(s))ds \quad \text{for } x \in B \quad \text{and } t \in J.$$

From the inequalities

$$\alpha = \alpha_1 + \alpha - \alpha_1 \le$$

$$\le \min_{t\in I} q(t) + \int_0^t m(t,s)ds \le\le q(t) + \int_0^t f(t,s,x(s))ds \le$$

$$\le \max_{t\in I} q(t) + \int_0^t M(t,s)ds \le\le \beta_1 + \beta - \beta_1 = \beta \quad \text{for} \quad x \in B,\ t \in J,$$

we infer that $G(B) \subset B$. Since

$$G(x)(\tau) - G(x)(t) =$$

$$= q(\tau) - q(t) + \int_0^\tau f(\tau,s,x(s))ds - \int_0^t f(t,s,x(s))ds =$$

$$= q(\tau) - q(t) + \int_0^t [f(\tau,s,x(s)) - f(t,s,x(s))]ds + \int_t^\tau f(\tau,s,x(s))ds,$$

$$q(\tau) - q(t) + \int_0^t r(\tau,t,s)ds + \int_t^\tau m(\tau,s)ds \le G(x)(\tau) - G(x)(t) \le$$

$$\le q(\tau) - q(t) + \int_0^t R(\tau,t,s)ds + \int_t^\tau M(\tau,s)ds$$

for fixed $t \in J$ and $\tau > t$ (analogously for fixed $t \in J$ and $\tau < t$). By the above inequalities we infer that the family $G(B)$ is equicontinuous at t. Since J is compact, $G(B)$ is equiuniformly continuous. In view of Ascoli's theorem it is relatively compact which proves (i).

Now, we verify that G is continuous. Let $x_0 \in B$ and let (x_n) be any sequence such that $x_n \in B$ for $n \in \mathbb{N}$ and $x_n \to x_0$. Fix $t \in J$. Put $\varphi_n(s) = f(t,s,x_n(s))$, $\varphi_0(s) = f(t,s,x_0(s))$ for $s \in [0,t]$. Obviously,

$\varphi_n(s) \to \varphi_0(s)$ for a.e. $s \in [0,t]$, as $n \to \infty$. Moreover $m(t,s) \leq \leq \varphi_n(s) \leq M(t,s)$ for $s \in [0,t]$. Hence by the well known dominated convergence theorem for (H-K)-integral (cf. [2]) we get $\lim_{n\to\infty} F(x_n)(t) = = F(x_0)(t)$. Hence $\lim_{n\to\infty} G(x_n)(t) = G(x_0)(t)$. In view of equiuniformly continuity of $G(B)$ we deduce that G is continuous which proves (ii). $\Diamond$

Remark. Consider the equation (1) with the Lebesgue integral instead of the (H-K)-one and the function h instead of f. Assume that the function $h : T \times [\alpha, \beta] \to \mathbb{R}$ is a Carathéodory function (see the assumption 2^0). Moreover, suppose that

a) there exist a function $\mu : T \to \mathbb{R}_+$ such that for every $t \in I$ the function $\mu(t, \cdot)$ is Lebesgue integrable on $[0,t]$ and
$$| h(t,s,x) | \leq \mu(t,s) \quad \text{for} \quad (t,s,x) \in T \times [\alpha, \beta];$$
b) there exists a function $\rho(\tau,t,s)$ $(0 \leq s \leq t \leq \tau \leq a)$ such that
$$| h(\tau,s,x) - h(t,s,x) | \leq \rho(\tau,t,s),$$
c) $\lim_{\tau - t \to 0_+} [(L)\int_0^t \rho(\tau,t,s)ds + (L)\int_t^\tau \mu(\tau,s)ds] = 0$ for fixed t or τ, where the sign "$(L)\int$" stands for the Lebesgue integral.

It is well known that under the above assumptions the equation (1) has at least one local continuous solution.

Now, let $f(t,s,x) = h(t,s,x) + m(t,s)$, $(t,s,x) \in T \times [\alpha,\beta]$, where h satisfies the assumptions a)-c) and $m : T \to \mathbb{R}$ is a function such that

(j) for every $t \in I$, $m(t,\cdot)$ is (H-K)-integrable;

(jj) $\lim_{\tau - t \to 0^+} \int_t^\tau m(\tau,s)ds = 0$ for fixed t or τ;

(jjj) $\lim_{\tau - t \to 0^+} \int_0^t (m(\tau,s) - m(t,s))ds = 0$ for fixed t or τ;

Obviously f satisfies $2^0 - 5^0$. But it is clear that h does not have to satisfy a)–c), m does not have to satisfy (j)–(jjj), in spite of this the sum $h + m = f$ will satisfy $2^0 - 5^0$.

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