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UNIQUENESS OF TWO-PARAMETER DYADIC MARTINGALES AND WALSH-FOURIER SERIES

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Abstract: We give the connection between quasi-measures, martingales and Walsh series and prove some uniqueness theorems for two-parameter dyadic martingales and Walsh series with respect to two types of almost everywhere convergence.

1. Introduction

It is well known that if a one-parameter Walsh series with coefficients tending to 0 converges to an integrable function, except possibly in a countable set, then that series is the Walsh-Fourier series of the limit function (see e.g. Crittenden, Shapiro [1]). The two-parameter analogue of this result can be found in Skvorcov [7] and Movsisjan [4].

Let $\mathbf{G}$ denote the dyadic group and S_{2^n} the 2^nth partial sum of a Walsh series S. Wade [8], [9] proved, that if

$$\lim_{n\to\infty} 2^{-n} S_{2^n}(x) = 0 \quad \text{for all} \quad x \in \mathbf{G},$$

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$$\lim_{n\to\infty} S_{2^n}(x) = f \quad \text{in measure}$$

for some $f \in L_1$ and

$$\limsup_{n\to\infty} |S_{2^n}(x)| < \infty,$$

except possibly in a countable set, then S is the Walsh-Fourier series of f.

In this paper we generalize this result for multi-parameter dyadic martingales and Walsh series. We consider the convergence in measure or a.e. convergence in Prigsheim sense and a.e. convergence taken over a cone. We follow basically the one-parameter proof in Schipp, Wade, Simon, Pál [6], but using martingale techniques we can extend and simplify the proof. Furthermore, we give the connection between quasi-measures, martingales and Walsh series.

2. The dyadic group and martingales

Let $\mathbf{Z}_2$ be the discrete cyclic group of order 2, i.e., the set $\{0, 1\}$ with the discrete topology and modulo 2 addition. Clearly, $\mathbf{Z}_2$ is a compact abelian group. The *dyadic group* $\mathbf{G}$ is defined to be the compact abelian group formed by taking the cartesian product of countably many copies of $\mathbf{Z}_2$, say

$$\mathbf{G} := \mathbf{Z}_2 \times \mathbf{Z}_2 \times \dots .$$

Thus $\mathbf{G}$ consists of sequences $x = (x_n, n \in \mathbb{N})$ where $x_n = 0$ or 1. The zero element of $\mathbf{G}$ is the sequence $0 := (x_n := 0,\ n \in \mathbb{N})$ and the group operation is given by

$$x + y := (|x_n - y_n|, n \in \mathbb{N})$$

for any $x = (x_n, n \in \mathbb{N})$ and $y = (y_n, n \in \mathbb{N})$ in $\mathbf{G}$.

Set $I_0(x) := \mathbf{G}$ for all $x \in \mathbf{G}$. For each $x \in \mathbf{G}$ and $n \in \mathbf{P} := \mathbb{N}\backslash\{0\}$ define

$$I_n(x) := \{y \in \mathbf{G} : y_i = x_i \quad \text{for} \quad 0 \le i < n\}.$$

We shall call these sets the *dyadic intervals* of $\mathbf{G}$. the dyadic intervals are evidently both open and closed.

Define a measure on $\mathbf{Z}_2$ by assigning each singleton the By definition

$$\mu(I_n(x)) = 2^{-n} \qquad (x \in \mathbf{G}, n \in \mathbb{N}).$$

It is easy to see that μ is the Haar measure on $\mathbf{G}$.

The functions

$$r_n(x) := (-1)^{x_n} \qquad (n \in \mathbb{N})$$

are called *Rademacher functions* and the product system generated by these functions is the *one-dimensional Walsh system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $0 \le n_k < 2$, $n_k \in \mathbb{N}$ and $x = (x_n, n \in \mathbb{N}) \in \mathbf{G}$.

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^2$ be its cartesian product $\mathbf{X} \times \mathbf{X}$ taken with itself. In this paper we investigate $\mathbf{G}^2$ and the product measure on it, which we denote again by μ. The cartesian product of two dyadic intervals is said to be a *dyadic rectangle.* For $n, m \in \mathbb{N}$ and $(x, y) \in \mathbf{G}^2$ set

$$I_{n,m}(x, y) := I_n(x) \times I_m(y).$$

The Kronecker product $(w_{n,m}; n, m \in \mathbb{N})$ of two Walsh systems is said to be the *two-dimensional Walsh system.* Thus

$$w_{n,m}(x, y) := w_n(x) w_m(y) \qquad ((x, y) \in \mathbf{G}^2).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,m}(x, y) : (x, y) \in \mathbf{G}^2\}$ will be denoted by $\mathcal{F}_{n,m}$ $(n, m \in \mathbb{N})$. It is easy to see that the σ-algebras $(\mathcal{F}_{n,m})$ are non-decreasing with respect to the usual partial ordering of $\mathbb{N}^2$. The conditional expectation operator relative to $\mathcal{F}_{n,m}$ $(n, m \in \mathbb{N})$ is denoted by $E_{n,m}$. We briefly write L_p instead of the real $L_p(\mathbf{G}^2, \mu)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{\mathbf{G}^2} |f|^p \, d\mu)^{1/p}$ $(0 < p \le \infty)$.

An integrable sequence $\mathbf{f} = (f_{n,m}; n, m \in \mathbb{N})$ is said to be a *martingale* if

(i) it is *adapted*, i.e. $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable for all $n, m \in \mathbb{N}$,
(ii) $E_{k,l} f_{n,m} = f_{k,l}$ for all $k \le n$ and $l \le m$.

The martingale $\mathbf{f} = (f_n; n, m \in \mathbb{N})$ is said to be L_p*-bounded* $(0 < p \le \infty)$ if $f_{n,m} \in L_p$ $(n, m \in \mathbb{N})$ and

$$\|\mathbf{f}\|_p := \sup_{n,m \in \mathbb{N}} \|f_{n,m}\|_p < \infty.$$

If $f \in L_1$ then it is easy to show that the sequence $(E_{n,m} f; n, m \in \mathbb{N})$ obtained from f is a martingale. Moreover, if $1 \le p < \infty$ and $f \in L_p$ then $(E_{n,m} f; n, m \in \mathbb{N})$ is L_p-bounded and

$$\lim_{n,m \to \infty} \|E_{n,m} f - f\|_p = 0,$$

consequently,

$$\|(E_{n,m} f; n, m \in \mathbb{N})\|_p = \|f\|_p$$

(see Neveu [5]). The converse of the last proposition holds also if $1 < p < \infty$ (see Neveu [5]): for an arbitrary martingale $\mathbf{f} = (f_{n,m}; n, m \in \mathbb{N})$ there exists a function $f \in L_p$ for which $f_{n,m} = E_{n,m} f$ if and only if f is L_p-bounded. If $p = 1$ then there exists a function $f \in L_1$ of the preceding type if and only if f is *uniformly integrable* (see Neveu [5]), namely, if

$$\lim_{c\to\infty} \sup_{n,m\in\mathbb{N}} \int_{\{|f_{n,m}|>c\}} |f_{n,m}| \, d\mu = 0.$$

Thus the map $f \mapsto (E_{n,m}f; n, m \in \mathbb{N}^2)$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p < \infty$ and from L_1 onto the space of uniformly integrable martingales. Every L_1-bounded martingale $(f_{n,m}; n, m \in \mathbb{N})$ converges in measure as $n, m \to \infty$ and a.e. as $n, m \to \infty$, $|n-m| \le \alpha$ $(\alpha \ge 0)$.

3. Quasi-measures and martingales

By a quasi-measure on $\mathbf{G}^2$ we mean a finitely additive real-valued set function defined on the dyadic rectangles of $\mathbf{G}^2$. The collection of quasi-measures on $\mathbf{G}^2$ will be denoted by $\mathbf{QM}(\mathbf{G}^2) = \mathbf{QM}$. The collection of finite Borel measures on $\mathbf{G}^2$ will be denoted by $\mathbf{M}(\mathbf{G}^2) = \mathbf{M}$.

A quasi-measure ν on $\mathbf{G}^2$ is said to belong to $\mathbf{M}$ if rectangle is both open and closed, ν must be countably additive on the collection of dyadic rectangles. Hence ν can be extended to a Borel measure on $\mathbf{G}^2$. In particular, if $\nu \in \mathbf{QM}$ is non-negative then ν belongs to $\mathbf{M}$.

For each $\nu \in \mathbf{QM}$ define the martingale $\mathbf{f}^\nu := (f^\nu_{n,m}; n, m \in \mathbb{N})$ by

(1) $$f^\nu_{n,m}(x,y) := 2^n 2^m \nu(I_{n,m}(x,y)) \qquad (n, m \in \mathbb{N}; x, y \in \mathbf{G}).$$

It is easy to see that the map $\nu \mapsto \mathbf{f}^\nu$ is a 1-1 linear map from $\mathbf{QM}$ onto the collection of martingales. For $\alpha \ge 0$ define the subset C_α of $\mathbb{N}^2$ by

$$C_\alpha := \{(n,m) \in \mathbb{N}^2 : |n-m| \le \alpha.\}$$

We will consider convergence over $\mathbb{N}^2$ and over C_α. Now we give the connection between quasi-measures and martingales.

Theorem 1. *We have*

(i) $\nu \in \mathbf{QM}$ *is of bounded variation* $\iff$ $\mathbf{f}^\nu$ *is* L_1*-bounded* $\iff$ $\iff$ $\nu \in \mathbf{M}$,

(ii) $\nu \in \mathbf{M}$ *is absolute continuous with respect to* μ $\iff$ $\mathbf{f}^\nu$ *is uniformly integrable,*

(iii) $\nu \in \mathbf{M}$ *is singular with respect to* μ $\iff$ $\mathbf{f}^\nu$ *is* L_1*-bounded and* $\lim_{n,m\to\infty} f^\nu_{n,m} = 0$ *in measure or* $\lim_{n,m\to\infty,(n,m)\in C_\alpha} f^\nu_{n,m} = 0$ *a.e.*

Proof. If $\nu \in \mathbf{QM}$ is of bounded variation then, by (1),

$$\int_{I_{n,m}(x,y)} |f^{\nu}_{n,m}|\, d\mu = |\nu(I_{n,m}(x,y))| \qquad (n,m \in \mathbb{N}; x,y \in \mathbf{G}).$$

From this it follows that $\|\mathbf{f}^{\nu}\|_1 \leq \|\nu\|$ where $\|\nu\|$ denotes the total variation of ν.

For the converse suppose that R_k $(k \in \mathbb{N})$ are disjoint dyadic rectangles. By the submartingale property of $(|f^{\nu}_{n,m}|; n,m \in \mathbb{N})$

$$\sum_{k=0}^{N} |\nu(R_k)| \leq \sum_{k=0}^{N} \int_{R_k} |f^{\nu}_{n,m}|\, d\mu \leq \|\mathbf{f}^{\nu}\|_1$$

where n and m are so big, such that R_k is $\mathcal{F}_{n,m}$ measurable for each $k = 0, \ldots, N$. Hence $\|\nu\| \leq \|\mathbf{f}^{\nu}\|_1$.

If $\nu \in \mathbf{M}$ then it has bounded variation, thus $\mathbf{f}^{\nu}$ is L_1-bounded. Conversely, if $\mathbf{f}^{\nu}$ is L_1-bounded, then it can be decomposed into the difference of two non-negative martingales (see e.g. Long [2] p.15 for one parameter, for two parameters the proof is similar). The corresponding two quasi-measures are non-negative and so they are Borel measures. Hence ν is also a Borel measure.

If $\nu \in \mathbf{M}$ is absolute continuous with Radon-Nikodym derivative $f \in L_1$ then $f^{\nu}_{n,m} = E_{n,m} f$ $(n,m \in \mathbb{N})$. The converse is also clear.

If $\nu \in \mathbf{M}$ is singular then $\nu(I_{n,m}(x,y)/\mu(I_{n,m}(x,y)$ converges to 0 a.e. as $n,m \to \infty$ and $(n,m) \in C_{\alpha}$ and, consequently, it converges to 0 in measure as $n,m \to \infty$. The proof of the theorem is complete. $\diamondsuit$

4. Uniqueness of martingales

A fundamental problem in the theory of martingales and Walsh series is the problem of uniqueness. That is, when is a given martingale or Walsh series the martingale obtained from an integrable function or the Walsh-Fourier series of an integrable function? Of course, this is true if the martingale or Walsh series converges in L_1 norm. We consider here the convergence in measure and the a.e. convergence.

Lemma 1. *Let $f \in L_1$, $(n_0, m_0) \in \mathbb{N}^2$, $(x_0, y_0) \in \mathbf{G}^2$ and M be a positive number. Let $\mathbf{f} = (f_{n,m}; n,m \in \mathbb{N})$ be a martingale, set $\mathbf{g} :=$ $= (g_{n,m}; n,m \in \mathbb{N})$ with*

$$g_{n,m} := f_{n,m} - E_{n,m} f \qquad ((n,m) \in \mathbb{N}^2),$$

and suppose $g_{n_0,m_0} \neq 0$ on $I_{n_0,m_0}(x_0,y_0)$. If $f_{n,m} \to f$ in mea-

sure as $n, m \to \infty$, *then there exists a dyadic rectangle* $I_{n,m}(x,y) \subset \subset I_{n_0,m_0}(x_0,y_0)$ *such that* $g_{n,m} \neq 0$ *on* $I_{n,m}(x,y)$ *and* $|f_{n,m}| > M$ *on* $I_{n,m}(x,y)$.

Proof. This is immediate if there is at least one point (u,v) in some $I_{n,m}(x,y) \subset I_{n_0,m_0}(x_0,y_0)$ such that $|f_{n,m}(u,v)| > M + |E_{n,m}f(u,v)|$.

Suppose to the contrary that

$$|f_{n,m}(u,v)| \leq M + |E_{n,m}f(u,v)|$$

for all $(u,v) \in I_{n_0,m_0}(x_0,y_0)$ and $n \geq n_0$, $m \geq m_0$. From this it follows that **f** and **g** are uniformly integrable. The martingale $(E_{n,m}f; n,m \in \in \mathbb{N})$ converges to f in measure and so $g_{n,m} \to 0$ in measure as $n,m \to \to \infty$. Since **g** is also convergent in L_1 norm, the limit must be 0 which means that

$$\lim_{n,m\to\infty} \int_{I_{n_0,m_0}(x_0,y_0)} g_{n,m}\, d\mu = 0.$$

However, since g_{n_0,m_0} is constant on $I_{n_0,m_0}(x_0,y_0)$, we have

$$\int_{I_{n_0,m_0}(x_0,y_0)} g_{n,m}\, d\mu =$$

$$= \int_{I_{n_0,m_0}(x_0,y_0)} g_{n_0,m_0}\, d\mu = 2^{-n_0-m_0} g_{n_0,m_0}(x_0,y_0) \neq 0$$

for all $n \geq n_0$ and $m \geq m_0$, which is a contradiction. $\Diamond$

Lemma 2. *The same statement is true if we suppose that* (n_0,m_0), $(n,m) \in C_\alpha$ *and* $f_{n,m} \to f$ *a.e. as* $n,m \to \infty$ *instead of the convergence in measure.*

We say that a martingale **f** satisfies the C-S *condition* if

$$\lim_{n,m\to\infty} 2^{-n-m} f_{n,m}(x,y) = 0$$

for all $(x,y) \in \mathbf{G}^2$.

Lemma 3. *Let* $(x_0,y_0) \in \mathbf{G}^2$, $(n_0,m_0) \in \mathbb{N}^2$ *and* **f** *be a martingale which satisfies the* C-S *condition. If* $f_{n_0,m_0} \neq 0$ *on* $I_{n_0,m_0}(x_0,y_0)$, *then there exists a dyadic rectangle* $I_{n,m}(x,y) \subset I_{n_0,m_0}(x_0,y_0)$ *such that* $(x_0,y_0) \notin I_{n,m}(x,y)$ *and* $f_{n,m} \neq 0$ *on* $I_{n,m}(x,y)$.

Proof. Suppose the lemma is false. For each $k \in \mathbb{N}$ set

$$I_k := I_{n_0+k,m_0+k}(x_0,y_0)$$

and $J_k := I_{k-1} \setminus I_k$. Since $(x_0,y_0) \notin J_k$, $f_{n_0+k,m_0+k} = 0$ on J_k $(k \in \mathbf{P})$. We show that

$$f_{n_0+k,m_0+k} = 2^{2k}\beta \quad on \quad I_k \qquad (k \in \mathbb{N}) \tag{2}$$

where $\beta := f_{n_0,m_0}(x_0,y_0)$. This is clear for $k = 0$. Suppose it is true for $k-1$. Then $f_{n_0+k-1,m_0+k-1} = 2^{2k-2}\beta$ on I_k and J_k. Thus

$$f_{n_0+k,m_0+k} - f_{n_0+k-1,m_0+k-1} = -2^{2k-2}\beta \quad on \quad J_k.$$

Since

$$E_{n_0+k-1,m_0+k-1}(f_{n_0+k,m_0+k} - f_{n_0+k-1,m_0+k-1}) = 0,$$

we conclude that

$$f_{n_0+k,m_0+k} - f_{n_0+k-1,m_0+k-1} = 3 \cdot 2^{2k-2}\beta \quad on \quad I_k,$$

which proves (2). In particular,

$$2^{-n-m} f_{n,m}(x_0, y_0) = 2^{-n_0-m_0}\beta$$

for all $n = n_0 + k$ and $m = m_0 + k$ ($k \in \mathbb{N}$). In other words $\mathbf{f}$ does not satisfy the C-S condition. This contradiction finishes the proof of the lemma. ◊

Lemma 4. *The same lemma holds if we take the limit over* C_α *in the definition of the* C-S *condition and if we suppose that* (n_0, m_0), $(n, m) \in \in C_\alpha$.

Now we can state our main result.

Theorem 2. *Suppose* E *is a countable subset of* $\mathbf{G}^2$ *and* $\mathbf{f}$ *is a martingale satisfying the* C-S *condition such that*

$$\limsup_{n,m\to\infty} |f_{n,m}(x, y)| < \infty \tag{3}$$

for all $(x, y) \in \mathbf{G}^2 \setminus E$. *If*

$$\lim_{n,m\to\infty} f_{n,m} = f \quad \textit{in measure} \tag{4}$$

for some function $f \in L_1$, *then* $\mathbf{f}$ *is the martingale obtained from* f.

Proof. Suppose the theorem is false. Then there exist $(x_0, y_0) \in \mathbf{G}^2$ and $(n_0, m_0) \in \mathbb{N}^2$ such that $f_{n_0,m_0} \neq E_{n_0,m_0} f$ on $I_{n_0,m_0}(x_0, y_0)$.

Set $E = \{(x_1, y_1), (x_2, y_2), \dots\}$ and $\mathbf{g} := \mathbf{f} - (E_{n,m}f; n, m \in \mathbb{N})$. By (1) and Th. 1(ii) the martingales $(E_{n,m}f; n, m \in \mathbb{N})$ and, consequently, $\mathbf{g}$ satisfy the C-S condition. It is easy to see there is a dyadic rectangle $J \in \mathcal{F}_{n',m'}$, $J \subset I_{n_0,m_0}(x_0, y_0)$ such that $(x_1, y_1) \notin J$ and $g_{n',m'}$ is non-zero on J. This is obvious if $(x_1, y_1) \notin I_{n_0,m_0}(x_0, y_0)$. If $(x_1, y_1) \in I_{n_0,m_0}(x_0, y_0)$ then use Lemma 3. Using Lemma 1 we choose a dyadic rectangle $I_1 \in \mathcal{F}_{n_1,m_1}$, $I_1 \subset J$ such that

$$|f_{n_1,m_1}| > 1 \quad \text{on} \quad I_1.$$

Applying Lemma 3 and Lemma 1 we can get dyadic rectangles $I_1 \supset \supset I_2 \supset \dots$ such that $(x_k, y_k) \notin I_k$ and pairs $(n_1, m_1), (n_2, m_2), \dots$ such that

$$|f_{n_k,m_k}| > k \quad \text{on} \quad I_k \qquad (k \in \mathbf{P}). \tag{5}$$

Since the dyadic rectangles are compact sets, there exists $(x, y) \in \cap_{k=1}^\infty I_k$. By construction, $(x, y) \notin E$. Hence by hypothesis,

$$\limsup_{n,m\to\infty} |f_{n,m}(x,y)| < \infty$$

which contradicts to (5). Therefore, $\mathbf{f}$ must be the martingale obtained from f. $\Diamond$

We formulate another version of this result.

Theorem 3. *If we change* (3) *and* (4) *to*

$$\limsup_{n,m\to\infty,(n,m)\in C_\alpha} |f_{n,m}(x,y)| < \infty$$

and

$$\lim_{n,m\to\infty,(n,m)\in C_\alpha} f_{n,m} = f \quad a.e.,$$

respectively, then the statement of Th. 2 *holds again.*

5. Uniqueness of Walsh series

Denote the (n,m)th partial sum of the formal two-parameter Walsh series

$$S := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{k,l} w_{k,l}$$

by

$$S_{n,m} := \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} c_{k,l} w_{k,l} \quad (n,m \in \mathbf{P}).$$

The Walsh-Fourier coefficients and the Walsh-Fourier series of an integrable function f are given by

$$\hat{f}(k,l) := \int_{\mathbf{G}^2} f w_{k,l}\, d\mu, \qquad Sf := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \hat{f}(k,l) w_{k,l},$$

respectively. We extend these definitions to quasi-measures as follows. For $\nu \in \mathbf{QM}$ define the Walsh-Fourier-Stieltjes coefficients and the Walsh-Fourier-Stieltjes series of ν by

$$\hat{\nu}(k,l) := \int_{\mathbf{G}^2} w_{k,l}\, d\nu, \qquad S\nu := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \hat{\nu}(k,l) w_{k,l},$$

respectively. It is easy to see that if ν is absolute continuous with Radon-Nikodym derivative f, then $\hat{\nu}(k,l) = \hat{f}(k,l)$ $(k,l \in \mathbb{N})$.

We can also introduce the Walsh-Fourier-Stieltjes series of martingales. If $\mathbf{f} = (f_{n,m}; n,m \in \mathbb{N})$ is a martingale then let

$$\hat{\mathbf{f}}(k,l) := \lim_{n,m\to\infty} \int_{\mathbf{G}^2} f_{n,m} w_{k,l}\, d\mu, \qquad S\mathbf{f} := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \hat{\mathbf{f}}(k,l) w_{k,l}.$$

Since $w_{k,l}$ is $\mathcal{F}_{n,m}$ measurable for large enough n and $m \in \mathbb{N}$, it can immediately be seen that this limit does exist. Note that if $f \in L_1$ then $E_{n,m}f \to f$ in L_1 norm as $n,m \to \infty$, hence

$$\hat{f}(k,l) = \lim_{n,m\to\infty} \int_{\mathbf{G}^2} (E_{n,m}f) w_{k,l}\, d\mu \qquad (k,l \in \mathbb{N}).$$

Thus the Walsh-Fourier-Stieltjes coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_{n,m}f)$ obtained from f. It is easy to prove that

$$S_{2^n,2^m}\mathbf{f} = f_{n,m} \qquad (n,m \in \mathbb{N})$$

for all martingales $\mathbf{f}$. Moreover, for $\nu \in \mathbf{QM}$ we have $\hat{\nu}(k,l) = \widehat{\mathbf{f}^{\nu}}(k,l)$ $(k,l \in \mathbb{N})$.

It follows from (1) and Th. 1 that the maps $\nu \mapsto S\nu$ and $\mathbf{f} \mapsto S\mathbf{f}$ are 1-1 linear maps from $\mathbf{QM}$ and from the set of martingales onto the collection of all Walsh series. We say that the Walsh series S satisfies the C-S condition if the martingale $(S_{2^n,2^m}; n,m \in \mathbb{N})$ satisfies it.

Now we can formulate Ths 2 and 3 for two-parameter Walsh series.

Theorem 4. *Suppose E is a countable subset of $\mathbf{G}^2$ and S is a Walsh series satisfying the* C-S *condition such that*

(6) $$\limsup_{n,m\to\infty} |S_{2^n,2^m}(x,y)| < \infty$$

for all $(x,y) \in \mathbf{G}^2 \setminus E$. If

(7) $$\lim_{n,m\to\infty} S_{2^n,2^m} = f \quad \textit{in measure}$$

for some function $f \in L_1$, then S is the Walsh-Fourier series of f.

Theorem 5. *If we change* (6) *and* (7) *to*

$$\limsup_{n,m\to\infty,(n,m)\in C_\alpha} |S_{2^n,2^m}(x,y)| < \infty$$

and

$$\lim_{n,m\to\infty,(n,m)\in C_\alpha} S_{2^n,2^m} = f \quad a.e.,$$

respectively, then the statement of Th. 4 *holds again.*

Remark. All the results can similarly be proved in the multi-parameter setting and for Vilenkin martingales and Vilenkin series.

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