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ON A GENERALIZATION OF A FORMULA OF SIERPINSKI

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Abstract: For fixed natural numbers $1 \le a \le b$ we consider $\rho_{a,b}(n)$ defined as the number of pairs $(u, v) \in \mathbb{N} \times \mathbb{Z}, u > |v|$ with $(u-v)^a(u+v)^b = n$. We prove a formula analogous to that of Sierpinski for differences of two squares, and an Ω_+ result for the remainder term in the asymptotic formula for the corresponding Dirichlet summatory function.

1. Introduction

For fixed natural numbers $1 \le a \le b$, let

$$\rho_{a,b}(n) = \#\{(u,v) \in \mathbb{N} \times \mathbb{Z}, u > |v| : (u-v)^a(u+v)^b = n\} \qquad (n \in \mathbb{N}).$$

To study the average order of this arithmetic function, we consider the Dirichlet summatory function

$$T_{a,b}(x) = \sum_{n \le x} \rho_{a,b}(n), \tag{1.1}$$

where x is a large real variable.

For the case $a = b = 1$, the question for the asymptotic behaviour of $T_{1,1}(x)$ is closely related to the classical divisor problem of Dirichlet, by the elementary formula, due to Sierpinski [11]

$$\rho_{1,1}(n) = d(n) - 2d\left(\frac{n}{2}\right) + 2d\left(\frac{n}{4}\right), \tag{1.2}$$

where $d(n)$ denotes the divisor function with $d(\cdot) = 0$ for non-integers.

For a historical survey on Dirichlet's divisor problem and the definition of the O- and the Ω-symbols see the textbook of Krätzel [5]. At present, it is known that

$$\sum_{n \le x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{1.3}$$

where γ denotes the Euler-Mascheroni constant and

$$\Delta(x) = O\Big(x^{23/73}(\log x)^{461/147}\Big), \tag{1.4}$$

due to Huxley [4].

Concerning lower estimates, it is known that

$$\Delta(x) = \Omega_+\Big((x \log x)^{1/4}(\log\log x)^{(3+2\log 2)/4} \exp\left(-B\sqrt{\log\log\log x}\right)\Big), \tag{1.5}$$

and

$$\Delta(x) = \Omega_-\Big(x^{1/4} \exp\left(c(\log\log x)^{1/4}(\log\log\log x)^{-3/4}\right)\Big), \tag{1.6}$$

for some positive constants B and c, established by Hafner [2], and Corrádi and Kátai [1], respectively.

For the special case $a = b = 1$, (1.2), (1.3), and (1.4) together yield,

$$T_{1,1}(x) = \frac{x}{2} \log x + (2\gamma - 1)\frac{x}{2} + \theta_{1,1}(x),$$

with

$$\theta_{1,1} = \Delta(x) - 2\Delta\left(\frac{x}{2}\right) + 2\Delta\left(\frac{x}{4}\right),$$

and therefore by (1.4)

$$\theta_{1,1} = O\Big(x^{23/73}(\log x)^{461/147}\Big).$$

Concerning lower estimates, the author proved in [7], [8], on the basis of (1.2), Ω- results for $\theta_{1,1}(x)$ which are as sharp as (1.5), resp. (1.6).

We show, that for the general case $(a, b) \neq (1, 1)$, there exists a formula quite analogous to (1.2), which is closely related to the asymmetric divisor function

$$d_{a,b}(n) = \sum_{u^a v^b = n} 1, \tag{1.7}$$

and to its corresponding Dirichlet summatory function

$$\sum_{n \le x} d_{a,b}(n) = \zeta\left(\frac{b}{a}\right) x^{1/a} + \zeta\left(\frac{a}{b}\right) x^{1/b} + \Delta_{a,b}(x). \tag{1.8}$$

The corresponding formula for the general case has the form

$$\rho_{a,b}(n) = d_{a,b}(n) - d_{a,b}\left(\frac{n}{2^a}\right) - d_{a,b}\left(\frac{n}{2^b}\right) + 2d_{a,b}\left(\frac{n}{2^{a+b}}\right). \tag{1.9}$$

A thorough account on the history of the asymmetric divisor problem and a survey on results concerning upper estimates for the remainder term $\Delta_{a,b}(x)$ is given in the textbook of Krätzel [5]. The today sharpest lower estimates were established by Hafner [3] and read

$$\begin{aligned} \Delta_{a,b}(x) &= \Omega_+\Big(x^\delta (\log x)^{a\delta} (\log\log x)^{(2\log 2 - 1)a\delta + 1} \\ &\qquad \exp\left(-B\sqrt{\log\log\log x}\,\right)\Big), \\ \Delta_{a,b}(x) &= \Omega_-\Big(x^\delta \exp\left(c(\log\log x)^{a\delta}(\log\log\log x)^{a\delta - 1}\right)\Big), \end{aligned} \tag{1.10}$$

where B, c are positive constants and

$$\delta = \frac{1}{2(a+b)}. \tag{1.11}$$

The objective of the present paper, is an Ω_+- result for the error term of (1.1), on the basis of (1.9), which is as sharp as (1.10).

Theorem. *For $1 \le a \le b$ natural numbers, and δ defined as in* (1.11), *we have*

$$T_{a,b}(x) = \frac{1}{2}\zeta\left(\frac{b}{a}\right) x^{1/a} + \frac{1}{2}\zeta\left(\frac{a}{b}\right) x^{1/b} + \theta_{a,b}(x),$$

with

$$\begin{aligned} \theta_{a,b}(x) &= \Omega_+\Big(x^\delta (\log x)^{a\delta} (\log\log x)^{(2\log 2 - 1)a\delta + 1} \\ &\qquad \exp\left(-B\sqrt{\log\log\log x}\,\right)\Big), \end{aligned}$$

where B is a positive constant.

2. Some results

Lemma 1. *Let $d_{a,b}(n)$ as in* (1.7). *If 2^{a+b} divides n we have*

$$d_{a,b}(n) - d_{a,b}\left(\frac{n}{2^a}\right) - d_{a,b}\left(\frac{n}{2^b}\right) + d_{a,b}\left(\frac{n}{2^{a+b}}\right) = 0.$$

Proof. Write $n = 2^{a+b+\alpha}u$, with u odd. Then

$$\begin{aligned} d_{a,b}(2^{a+b+\alpha}) &= \sum_{u^a v^b = 2^{a+b+\alpha}} 1 = \\ &= \sum_{\substack{u^a v^b = 2^{a+b+\alpha} \\ 2\mid u}} 1 + \sum_{\substack{u^a v^b = 2^{a+b+\alpha} \\ 2\mid v}} 1 - \sum_{\substack{u^a v^b = 2^{a+b+\alpha} \\ 2\mid u, 2\mid v}} 1 = \\ &= \sum_{u^a v^b = 2^{b+\alpha}} 1 + \sum_{u^a v^b = 2^{a+\alpha}} 1 - \sum_{u^a v^b = 2^{\alpha}} 1 = \\ &= d_{a,b}(2^{b+\alpha}) + d_{a,b}(2^{a+\alpha}) - d_{a,b}(2^{\alpha}). \end{aligned}$$

The proof follows from the multiplicativity of $d_{a,b}(\cdot)$. ◊

Proposition 1 (Generalized Sierpinski identity). *Define*

$$\rho_{a,b}(n) = \#\{(u,v) \in \mathbb{N} \times \mathbb{Z}, u > |v| : (u-v)^a (u+v)^b = n\},$$

then

$$\rho_{a,b}(n) = d_{a,b}(n) - d_{a,b}\left(\frac{n}{2^a}\right) - d_{a,b}\left(\frac{n}{2^b}\right) + 2d_{a,b}\left(\frac{n}{2^{a+b}}\right), \tag{2.1}$$

where $d_{a,b}(x) = 0$ *for* $x \notin \mathbb{N}$.

Proof. Let

$$X(n) = \{(x,y) \in \mathbb{N} \times \mathbb{Z}, x > |y| : (x-y)^a (x+y)^b = n\},$$

$$Y(n) = \{(u,v) \in \mathbb{N}^2 : u^a v^b = n\}.$$

Obviously $|X(n)| = \rho_{a,b}(n)$ and $|Y(n)| = d_{a,b}(n)$.

Case 1: n is odd.

$$(x,y) \in X(n) \leftrightarrow (u,v) = (x-y, x+y) \in Y(n)$$

defines a bijection between $X(n)$ and $Y(n)$. Therefore $\rho_{a,b}(n) = d_{a,b}(n)$.

Case 2: 2^{a+b} divides n.

$$(x,y) \in X(n) \leftrightarrow (u,v) = \left(\frac{x-y}{2}, \frac{x+y}{2}\right) \in Y\left(\frac{n}{2^{a+b}}\right)$$

defines a bijection between $X(n)$ and $Y(\frac{n}{2^{a+b}})$; therefore $\rho_{a,b}(n) = d_{a,b}(\frac{n}{2^{a+b}})$ which equals the right-hand side of (2.1), by Lemma 1.

Case 3: n is even and $n \not\equiv 0 \pmod{2^{a+b}}$. In this case it is easily seen that both sides of (2.1) are zero. ◊

Let a be as above. We say that n is *a-full* if for any prime q which divides n, q^a divides n too. For large positive real x we define $A(x)$

as the set of "a-full", positive numbers less than x. It is known that $\# A(x) \ll x^{1/a}$ (see e.g. Krätzel [5], p. 276). We put

$$H(x) = \{n \in A(x) \mid w(n) \geq 2\log\log x - B\sqrt{\log\log x}\},$$

where B is a positive constant and $\omega(n)$ is the number of distinct prime divisors of n.

Lemma 2.

$$\#H(x) \ll x^{1/a}(\log x)^{1-2\log 2}\exp\left(B\log 2\sqrt{\log\log x}\right).$$

Proof. By the definition of $H(x)$, we have

$$\#H(x)(\log x)^{2\log 2}\exp\left(-B\log 2\sqrt{\log\log x}\right) \leq \sum_{n\in A(x)} 2^{w(n)}.$$

To determine the asymptotic behaviour of this last sum, define

$$f(n) = \begin{cases} 2^{\omega(n)} & \text{if n } a\text{-full}, \\ 0 & \text{else.} \end{cases}$$

Clearly $f(n)$ is multiplicative. Consider the generating function

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \prod_{p \text{ prime}}\left(1 + \frac{2}{p^{as}} + \frac{2}{p^{(a+1)s}} + \ldots\right) = \zeta^2(as)G(s),$$

where $G(s)$ has a Dirichlet series absolutely convergent for Re $s > \frac{1}{a+1}$. By standard techniques Lemma 2 follows. ◊

Lemma 3. *For δ defined as in* (1.11), *and for any integer m, we have*

$$\sum_{\substack{n\leq x\\(n,m)=1}}\frac{\tau_{a,b}(n)}{n^{1-\delta}} = K\prod_{\substack{q|m\\q\ prime}}\left(1-\frac{1}{q}\right)^2 x^{\delta}\log x + O(x^{\delta})$$

where

(2.2) $$\tau_{a,b}(n) \overset{\text{def}}{=} \sum_{u^a v^b = n} u^{a-1}v^{b-1},$$

$K = (ab\delta)^{-1}$ *and the implied O-constant depends on a and b, but not on m.*

Proof. Write

(2.2) $$\frac{\tau_{a,b}(n)}{n} = \lambda(n) \quad \text{with} \quad \lambda(n) = \sum_{u^a v^b = n}\frac{1}{uv}.$$

Since the generating function

$$\sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{\lambda(n)\, n^{\delta}}{n^{s}} = \prod_{i=1}^{2} \sum_{d|m} \mu(d) d^{a_i(\delta-s)-1} \zeta(a_i(s-\delta)+1),$$

$$(a_1 = a, a_2 = b)\,, \quad (\operatorname{Re} s \geq \delta)$$

has a pole of order 2 at $s = \delta$, the result follows by standard techniques. ◊

Lemma 4. *For $\lambda(n)$ as above,*

$$\sum_{n\leq x} \lambda(n)\, n^{\delta} (w(n) - 2\log\log x)^2 = O(x^{\delta} \log x \log\log x).$$

Proof. Let q and r denote prime numbers. First we observe that

$$\sum_{n\leq x} \lambda(n)\, n^{\delta} w(n) = \sum_{n\leq x} \lambda(n)\, n^{\delta} \sum_{q^a|n} 1 = \sum_{q^a\leq x} \sum_{l\leq \frac{x}{q^a}} \lambda(q^a l) q^{a\delta} l^{\delta} =$$

$$= \sum_{q^a\leq x} \sum_{\alpha\geq 0} \lambda(q^{a+\alpha}) q^{(a+\alpha)\delta} \sum_{\substack{k\leq \frac{x}{q^{a+\alpha}}\\(k,q)=1}} \lambda(k) k^{\delta} =$$

$$= \sum_{q^a\leq x} \sum_{\alpha\geq 0} \lambda(q^{a+\alpha}) \Big\{ K\Big(1-\frac{1}{q}\Big)^2 x^{\delta} \log x + O(x) \Big\} =$$

$$= 2\, K x^{\delta} \log x \log\log x + O(x^{\delta} \log x).$$

(Use Lemma 3 and $\sum_{\alpha\geq 0} \lambda(q^{a+\alpha}) = \frac{2}{q} + O(\frac{1}{q^2})$.)

In the same way we conclude that

$$\sum_{n\leq x} \lambda(n)\, n^{\delta} \omega(n) \omega(n) = \sum_{n\leq x} \lambda(n)\, n^{\delta} \sum_{q^a, r^a|n} 1 =$$

$$= \sum_{q^a\leq x} \sum_{r^a\leq x} \sum_{n\leq x} \lambda(n)\, n^{\delta} =$$

$$= \Big\{ \sum_{\substack{q^a, r^a\leq x\\ q\neq r}} + \sum_{\substack{q^a, r^a\leq x\\ q=r}} \Big\} \sum_{n\leq x} \lambda(n)\, n^{\delta}.$$

Obviously,

$$\sum_{\substack{q^a, r^a\leq x\\ q=r}} \sum_{n\leq x} \lambda(n)\, n^{\delta} \ll x^{\delta} \log x \log\log x\,.$$

The main term is given by

$$\sum_{\substack{q^a, r^a \le x \\ q \ne r}} \sum_{k \le \frac{x}{q^a r^a}} \lambda(kq^a r^a) k^\delta q^{a\delta} r^{a\delta} =$$

$$= \sum_{\substack{q^a, r^a \le x \\ q \ne r}} \sum_{\alpha \ge 0} \sum_{\beta \ge 0} \lambda(q^{a+\alpha}) q^{(a+\alpha)\delta} \lambda(r^{a+\beta}) r^{(a+\beta)\delta} \sum_{\substack{l \le \frac{x}{q^{a+\alpha} r^{a+\beta}} \\ (l, qr)=1}} \lambda(l) l^\delta =$$

$$= \sum_{\substack{q^a, r^a \le x \\ q \ne r}} \sum_{\alpha \ge 0} \sum_{\beta \ge 0} \lambda(q^{a+\alpha}) \lambda(r^{a+\beta}) K \Big(\Big(1 - \frac{1}{q}\Big)\Big(1 - \frac{1}{r}\Big)\Big)^2 x^\delta \log x +$$

$$+ O(x^\delta) =$$

$$= K\, x^\delta \log x \Big(\sum_{q^a \le x} \sum_{\alpha \ge 0} \lambda(q^{a+\alpha}) \Big)^2 + O(x^\delta \log x \log\log x) =$$

$$= 4\, K x^\delta \log x (\log\log x)^2 + O(x^\delta \log x \log\log x) \; .$$

From these two formulas Lemma 4 follows. ◊

Lemma 5.

$$\sum_{\substack{n \notin H(x) \\ n \le x}} \lambda(n)\, n^\delta \ll \frac{1}{A^2} x^\delta \log x \, .$$

Proof. This is an immediate consequence of Lemma 4. ◊

Lemma 6.

$$\sum_{n \le x} \frac{\tau_{a,b}(n)}{n^{1-\delta}} \Big(1 - \sqrt{\frac{n}{x}}\Big) = \frac{2}{3ab\delta} x^\delta \log x + O(x^\delta).$$

Proof. We use

$$\sum_{n \le x} \tau_{a,b}(n) = \frac{1}{ab} x \log x + O(x)$$

(see e.g. Krätzel [5], p. 211, Lemma 5.6) and partial summation. ◊

Lemma 7 (Dirichlet's approximation principle). *Let* $\underline{\alpha} = (\alpha_1, \dots, \alpha_s) \in \in \mathbb{R}^s$, $q \in \mathbb{N}$, $t_0 \in \mathbb{R}^+$, *then there exists* $t \in \mathbb{R}$ *with* $||t\underline{\alpha}|| < \frac{1}{q}$ *and* $t_0 < t < t_0 q^s$, *where* $||.||$ *denotes the distance from the nearest integer.*

3. Proof of the theorem

We start from formulas (47), (48) of Krätzel [6], with a slight change of notation: For $x \ge 0$, we have

$$\Delta_1(x) \overset{\text{def}}{=} \int_1^x \left(\Delta_{a,b}(t) - \frac{1}{4}\right) dt = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^2} F(nx), \tag{3.1}$$

where

$$F(w) = w\rho_0(w) - \int_k^w \rho_0(u)\, du = O(w^{1-\delta}), \tag{3.2}$$

with

$$\rho_0(u) = \int_0^\infty \sin 2\pi t^{1/b} \sin 2\pi (\frac{x}{t})^{1/a} \frac{dt}{t},$$

and

$$\rho_0'(u) = c_1 \sin\left(2\pi c_2 u^{\frac{1}{a+b}} + \frac{\pi}{4}\right) + O\left(u^{-\delta-1}\right), \tag{3.3}$$

the asymptotic expansion in (3.3) taken from Nowak [9], formula (2.18). From (3.2), we see that the sum in (3.1) converges absolutely and uniformly on every compact set. By (3.2) and (3.3), $F'(w)$ has the asymptotic expansion

$$F'(w) = c_3 w^\delta \sin\left(2\pi c_2 w^{2\delta} + \frac{\pi}{4}\right) + O(w^{-\delta}), \tag{3.4}$$

where c_3, c_2 are computable positive constants depending on a and b. From (2.1) and (3.1) it is easy to see that

$$\begin{aligned} \theta_1(x) &:= \int_1^x (\theta_{a,b}(t) - \frac{1}{4})\, dt \\ &= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n^2} \Big\{F(nx)- \\ &\quad - 2^a F\left(\frac{n}{2^a}x\right) - 2^b F\left(\frac{n}{2^b}x\right) + 2^{a+b+1} F(\frac{n}{2^{a+b}}x)\Big\}. \end{aligned} \tag{3.5}$$

Let $g \in C^1[a,b]$. Multiply (3.5) by $g'(t)$, integrate term by term, (the series being absolutely and uniformly convergent), integrate by parts each term and use (3.5) again with $x = a$, resp. $x = b$. This yields,

$$\begin{aligned} \int_a^b g(t) \left(\theta_{a,b}(t) - \frac{1}{4}\right) dt &= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)}{n} \int_a^b g(t) \Big\{F'(nx)- \\ &\quad - F'\left(\frac{n}{2^a}x\right) - F'\left(\frac{n}{2^b}x\right) + 2F'\left(\frac{n}{2^{a+b}}x\right)\Big\}. \end{aligned} \tag{3.6}$$

For large positive real x, we define

$$K(y) = \frac{1}{2\pi}\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 \quad \text{and} \quad K_x(y) = \lambda_x\, K(\lambda_x y),$$

with $\lambda_x = 2\pi c_2 x^{2\delta}$, δ defined as in (1.11). It is well known (cf. e.g. [10, p. 19]) that

$$\int_{-\infty}^{\infty} \mathrm{e}^{iay}\, K(y)\, dy = \begin{cases} 1-|a|, & |a|<1 \\ 0, & |a|\geq 1. \end{cases}$$

By observing that $K_x'(u) = O(u^{-2})$ for $u \geq 1$, (uniformly for x), and applying integration by parts, we conclude that

$$\int_{-1}^{1} \mathrm{e}^{iau}\, K_x(u)\, du = \begin{cases} 1-\frac{a}{\lambda_x}+O(\frac{1}{a}), & 0<a\leq\lambda_x \\ O(\frac{1}{a}), & \text{else}\ . \end{cases}$$

Then by a suitable choice of $g(t)$, a, b in (3.4), and the substitutions $t = u^2$ and $u \to u + T$, we get from (3.4) and (3.6)

$$J(T) := \pi^2 c_1 \int_{-1}^{1} K_x(u) \frac{\theta_{a,b}(u+T)^{1/2\delta} - \frac{1}{4}}{(u+T)^{2\delta}}\, du =$$

$$= \sum_{n\leq x} a_n \left(1-\frac{\lambda_n}{\lambda_x}\right) \sin\left(\lambda_n T + \frac{\pi}{4}\right) +$$

$$+ \sqrt{2} \sum_{n\leq 2^{a+b}x} a_n \left(1-\frac{\mu_n}{\lambda_x}\right) \sin\left(\mu_n T + \frac{\pi}{4}\right) -$$

$$- \frac{1}{2^{a\delta}} \sum_{n\leq 2^{a}x} a_n \left(1-\frac{\nu_n}{\lambda_x}\right) \sin\left(\nu_n T + \frac{\pi}{4}\right) -$$

$$- \frac{1}{2^{b\delta}} \sum_{n\leq 2^{b}x} a_n \left(1-\frac{\eta_n}{\lambda_x}\right) \sin\left(\eta_n T + \frac{\pi}{4}\right) + O(1)\,,$$

where $\lambda_n = 2\pi c_2 n^{2\delta}$, $\mu_n = \pi c_2 n^{2\delta}$, $\nu_n = 2^{1-2a\delta}\pi c_2 n^{2\delta}$, $\eta_n = 2^{1-2b\delta} c_2 n^{2\delta}$, and $a_n = \tau_{a,b}(n) n^{\delta-1}$ for short, throughout the rest of the paper and δ defined as in (1.11).

We decompose this representation in order to apply Lemma 7, observing that $\tau_{a,b}(n) = 0$ for n not a-full, and estimate the other sums trivially by $-1 \leq \sin(\cdot) \leq 1$, to get

$$J(T) \geq \sum_{n \in H(x)} a_n \left(1 - \frac{\lambda_n}{\lambda_x}\right) \sin\left(\lambda_n T + \frac{\pi}{4}\right) +$$

$$+ \sqrt{2} \sum_{n \in H(2^{a+b}x)} a_n \left(1 - \frac{\mu_n}{\lambda_x}\right) \sin\left(\mu_n T + \frac{\pi}{4}\right) -$$

$$- 2^{-a\delta} \sum_{n \leq 2^a x} a_n \left(1 - \frac{\nu_n}{\lambda_x}\right) - 2^{-b\delta} \sum_{n \leq 2^b x} a_n \left(1 - \frac{\eta_n}{\lambda_x}\right) -$$

$$- \sum_{\substack{n \notin H(x) \\ n \leq x}} a_n \left(1 - \frac{\lambda_n}{\lambda_x}\right) - \sum_{\substack{n \notin H(2^{a+b}x) \\ n \leq 2^{a+b}x}} a_n \left(1 - \frac{\mu_n}{\lambda_x}\right).$$

The contribution from the last two sums is

$$\ll \frac{1}{B^2} x^\delta \log x,$$

by Lemma 5 with B a parameter at our disposition, whereas the contributions of the third and fourth sum can be estimated by Lemma 6.

The key step is now the application of Dirichlet's approximation principle to the first and the second sum. We apply this principle to those terms in the sums which yield the main contribution. To this end let $N_1 = \#H(x)$, and $N_2 = \#H(2^{a+b}x)$. For any $T_0 \geq 1$, we apply Lemma 7 to $\underline{\alpha} = (\lambda'_1, \cdots, \lambda'_{N_1}, \mu'_1, \cdots, \mu'_{N_2}) \in \mathbb{R}^{N_1+N_2}$ where $\lambda_n = 2\pi c_2 n^{2\delta} = 2\pi\lambda'_n$ and $\mu_n = \pi c_2 n^{2\delta} = 2\pi\mu'_n$, to find a T in the interval

$$T_0 < T < T_0 4^{q(N_1+N_2)} \tag{3.7}$$

with $\sin(\lambda_n T + \frac{\pi}{4}) \geq c > 0$ for all $n \in H(x)$ and $\sin(\mu_n T + \frac{\pi}{4}) \geq c$ for all $n \in H(2^{a+b}x)$ $(c = \sin\{\frac{\pi}{4}(1 - \frac{1}{q})\})$.

Therefore,

$$\begin{aligned} J(T) &\geq \frac{2}{3ab\delta}\left(3c - 2 - \frac{3}{B^2}\right) x^\delta \log x \\ &\geq C x^\delta \log x \end{aligned}$$

by a suitable choice of q and B. By a short calculation we derive from (3.7)

$$\log T \ll \log T_0 + N_1 + N_2$$

and by Lemma 2

$$J(T) \gg (\log T)^{a\delta}(\log\log T)^{(2\log 2-1)a\delta+1}\exp\left(-B\log 2\sqrt{\log\log\log T}\,\right).$$

Since $K_x(u)$ is positive and

$$0 < b_1 \le \int_{-1}^{1} K_x(u)\,du \le 1$$

with an absolute constant b, uniformly in $x \ge 1$, we may conclude that there exists a value v with $T-1 \le v \le T+1$ for which

$$\frac{\theta_{a,b}(u+T)^{1/2\delta}-\frac{1}{4}}{(u+T)^{2\delta}} \gg$$

$$\gg (\log v)^{a\delta}(\log\log v)^{(2\log 2-1)a\delta+1}\exp\left(-B\log 2\sqrt{\log\log\log v}\right).$$

Since $v > T_0 - 1$, and T_0 can be chosen arbitrarily large, this completes the proof of the theorem. $\Diamond$

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