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CONVOLUTIONS OF CERTAIN CLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract: The author establishes certain results concerning convolution or Hadamard product of meromorphic univalent functions with positive coefficients. The study demonstrates, in some respects, properties analogous to those possessed by the corresponding classes of univalent analytic function with negative coefficients. It is interesting to note that our results are valid for the usual "Hadamard product" while "quasi-Hadamard product" has been used to study analogous properties for univalent analytic functions with negative coefficients.

1. Introduction

Let T denote the class of functios of the form

$$\phi(z) = z - \sum_{n=2}^{\infty} c_n z^n, \quad c_n \geq 0$$

that are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. A function $f \in T$ is said to be in the class $T^*(A, B)$ if $z\phi'(z)/\phi(z)$ has a representation of the form

$$z\frac{\phi'(z)}{\phi(z)}=\frac{1+A\omega(z)}{1+B\omega(z)}$$

where $\omega(z)$ is analytic and $|\omega(z)|\le|z|$ in the unit disc U; A and B are real constants satisfying $-1\le A<B\le 1$. Let $K(A,B)$ be the class of functions $\phi\in T$ such that $z\phi'(z)\in T^*(A,B)$.

The quasi-Hadamard product of two functions $\phi(z)=z-\sum_{n=2}^{\infty}c_nz^n$ and $\psi(z)=z-\sum_{n=2}^{\infty}d_nz^n$ with $c_n,d_n\ge 0$ is defined by

$$\phi(z)*\psi(z)=z-\sum_{n=2}^{\infty}c_nd_nz^n.$$

Recently, Padmanabhan and Ganesan [2] have obtained several properties concerning quasi-Hadamard product for the classes $T*(A,B)$ and $K(A,B)$ extending the corresponding results of Schild and Silverman [3].

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$$

that are regular in $D=\{z:0<|z|<1\}$, having a simple pole at the origin. A function $f\in\Sigma$ is said to be in the class $\Sigma(A,B)$ if $zf'(z)/f(z)$ has a representation of the form

$$z\frac{f'(z)}{f(z)}=-\frac{1+A\omega(z)}{1+B\omega(z)}$$

where $\omega(z)$ is analytic and $|\omega(z)|\le|z|$ in the unit disc U; A and B are real constants satisfying $-1\le A<B\le 1$. Let $C(A,B)$ be the class of functions $f\in\Sigma$ such that $-zf'(z)\in\Sigma(A,B)$.

Denote by Σ_p the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,\qquad a_n\ge 0$$

that are regular and univalent in D and let

$$\Sigma_p(A,B)=\Sigma_p\cap\Sigma(A,B)\quad\text{and}\quad C_p(A,B)=\Sigma_p\cap C(A,B).$$

Let us define the convolution or Hadamard product of two meromorphic functions $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$ and $g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n$ with $a_n,b_n\ge 0$ by

$$f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Since to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, the aim of the present paper is to establish certain results concerning the convolution of meromorphic univalent functions analogous to Padmanabhan and Ganesan [2]. The results thus obtained generalize the corresponding results concerning convolution of meromorphic starlike functions of order $\alpha (0 \le \alpha < 1)$ obtained by Juneja and Reddy [1]. It is interesting to note that our results are valid for the usual "Hadamard product" while Padmanabhan and Ganesan [2] used in their results "quasi-Hadamard product" instead of the usual "Hadamard product".

In the sequel we make use of the following lemmas which can easily be proved using similar arguments as given by Juneja and Reddy [1].

Lemma 1. *A function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \ge 0$ *is in* $\Sigma_p(A, B)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{n(B+1) + A + 1}{B - A} a_n \le 1.$$

Lemma 2. *A function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \ge 0$ *is in* $C_p(A, B)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{n\{n(B+1) + A + 1\}}{B - A} a_n \le 1.$$

2. Convolution properties of functions in $\Sigma_p(A, B)$ and $C_p(A, B)$

Theorem 1. *If* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$, $a_n, b_n \ge 0$ *are elements of* $\Sigma_p(A, B)$, *then the Hadamard product* $f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$ *is an element of* $\Sigma_p(A_1, B_1)$ *with* $-1 \le A < B \le 1$ *where* $A_1 \le 1 - 2k$, $B_1 \ge (A_1 + k)/(1 - k)$ *with*

$$k = \frac{2(B-A)^2}{(B+A+2)^2 + (B-A)^2}$$

and that the bounds for A_1 *and* B_1 *cannot be improved.*

Proof. Suppose $f(z)$ and $g(z)$ are in $\Sigma_p(A,B)$. In view of Lemma 1 we have

$$\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} a_n \leq 1 \tag{1}$$

and

$$\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} b_n \leq 1. \tag{2}$$

We wish to find values of A_1, B_1 such that $-1 \leq A_1 < B_1 \leq 1$ for which $f(z) * g(z) \in \Sigma_p(A_1, B_1)$. Equivalently we want to determine A_1, B_1 satisfying

$$\sum_{n=1}^{\infty} \frac{n(B_1+1)+A_1+1}{B_1-A_1} a_n b_n \leq 1. \tag{3}$$

Using the Couchy-Schwarz inequality together with (1) and (2) we get

$$\sum_{n=1}^{\infty} u\sqrt{a_n b_n} \leq \left(\sum_{n=1}^{\infty} u a_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} u b_n\right)^{\frac{1}{2}} \leq 1 \tag{4}$$

where

$$u = \frac{n(B+1)+A+1}{B-A}.$$

The inequality (3) is satisfied if

$$u_1 a_n b_n \leq u\sqrt{a_n b_n}$$

where

$$u_1 = \frac{n(B_1+1)+A_1+1}{B_1-A_1}.$$

for $n \geq 1$, that is if

$$u_1 \sqrt{a_n b_n} \leq u.$$

But from (4) we have

$$\sqrt{a_n b_n} \leq \frac{1}{u}, \quad u \geq 1.$$

Thus it is enough to find u_1 such that

$$u_1 \leq u^2. \tag{5}$$

The inequality (5) is equivalent to

$$\frac{n(B_1+1)+A_1+1}{B_1-A_1} \leq \left(\frac{n(B+1)+A+1}{B-A}\right)^2 = u^2, \quad n \geq 1.$$

This yields

$$A_1 \leq \frac{u^2 B_1 - n(B_1+1) - 1}{u^2+1}. \tag{6}$$

Now (6) gives on simplification

(7) $$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{n+1}{u^2+1}, \quad n \geq 1.$$

The right-hand member of (7) decreases as n increases and is maximum for $n = 1$. Thus (7) is satisfied provided

(8) $$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{2(B-A)^2}{(B+A+2)^2 + (B-A)^2} = k, \quad \text{say.}$$

Obviously $k < 1$ and fixing A_1 in (8), we get

$$B_1 \geq \frac{k + A_1}{1 - k}.$$

It is easy to verify that $-1 < A_1 < B_1 \leq 1$. If we take

$$f(z) = g(z) = \frac{1}{z} + \frac{B-A}{B+A+2} z \in \Sigma_p(A, B)$$

we see that

$$f(z) * g(z) = \frac{1}{z} + \frac{(B-A)^2}{(B+A+2)^2} z.$$

Then

$$\frac{B_1 + A_1 + 2}{B_1 - A_1} = \frac{(B+A+2)^2}{(B-A)^2}.$$

Showing that $f(z) * g(z) \in \Sigma_p(1 - 2k, 1)$, with k as in (8). ◊

Remark. Putting $A = 2\alpha - 1$, $B = 1$, the class $\Sigma_p(A, B)$ reduces to the class $\Sigma_p(\alpha)$ of meromorphic starlike functions of order α, $0 \leq \alpha < 1$ with positive coefficients and our result yields the corresponding result obtained by Juneja and Reddy [1] on the convolution of two functions in $\Sigma_p(\alpha)$.

Corollary. *Let $f(z)$ and $g(z)$ as in* Th. 1. *Then the function*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \sqrt{a_n b_n} z^n \in \Sigma_p(A, B).$$

Proof. The result follows immediately from (4) using the Cauchy-Schwarz inequality. For the same functions as in Th. 1, the result is best possible. ◊

Theorem 2. *If $f(z) \in \Sigma_p(A, B)$ and $g(z) \in \Sigma_p(A', B')$, then $f(z) * * g(z) \in \Sigma_p(A_1, B_1)$ where $A_1 \leq 1 - 2k$ and $B_1 \geq \frac{A_1+1}{1-k}$ with*

$$k = \frac{2(B-A)(B'-A')}{(B+A+2)(B'+A'+2) + (B-A)(B'-A')}.$$

The result is best possible.

Proof. Proceeding exactly as in Th. 1, we require to show that

$$\frac{n(B_1+1)+A_1+1}{B_1-A_1} \leq$$
$$\leq \left(\frac{n(B+1)+A+1}{B-A}\right)\left(\frac{n(B'+1)+A'+1}{B'-A'}\right)=C$$

for all $n \geq 1$. This on simplification yields

$$\frac{B_1-A_1}{B_1+1} \geq \frac{n+1}{C+1}. \tag{9}$$

The function $(n+1)/(C+1)$ is decreasing with respect to n and is maximum for $n=1$. Thus (9) is satisfied provided

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(B-A)(B'-A')}{(B+A+2)(B'+A'+2)+(B-A)(B'-A')}=k. \tag{10}$$

Clearly $k<1$. Fixing A_1 in (10) we get $B_1 \geq \frac{A_1+k}{1-k}$. As we require $B_1 \leq 1$, we immediately obtain $A_1 \leq 1-2k$.

It is easily seen that the result is best possible for the functions

$$f(z)=\frac{1}{z}+\frac{B-A}{B+A+2}z \in \Sigma_p(A,B)$$
$$g(z)=\frac{1}{z}+\frac{B'-A'}{B'+A'+2}z \in \Sigma_p(A',B'). \qquad \Diamond$$

Remark. Taking $A=2\alpha-1$, $B=1$, $A'=2\gamma-1$, and $B'=1$ in Th. 2 we get the corresponding result obtained by Juneja and Reddy [1].

Corollary. *If $f(z), g(z), h(z) \in \Sigma_p(A,B)$ then $f(z)*g(z)*h(z) \in \Sigma_p(A_2,B_2)$ where $A_2 \leq 1-2k_1$ and $B_2 \geq \frac{A_2+k_1}{1-k_1}$ with*

$$k_1=\frac{2(B-A)(B_1-A_1)}{(B+A+2)(B_1+A_1+2)+(B-A)(B_1-A_1)}$$

where A_1, B_1 are as given in Th. 1.

Proof. Since $f(z), g(z) \in \Sigma_p(A,B)$ by Th. 1, $f(z)*g(z) \in \Sigma_p(A_1,B_1)$ where $A_1 \leq 1-2k_1$ and $B_1 \geq \frac{A_1+k}{1-k}$ with

$$k=\frac{2(B-A)^2}{(B+A+2)^2+(B-A)^2}.$$

Now letting $f(z) \in \Sigma_p(A_1,B_1)$ and $h(z) \in \Sigma_p(A,B)$, the result follows by Th. 2. $\Diamond$

Using Lemma 2 and proceeding exactly as in Th. 1 we have the following result for functions of the class $C_p(A,B)$.

Theorem 3. *If $f(z) \in C_p(A,B)$ and $g(z) \in C_p(A',B')$ then $f(z)*g(z) \in C_p(A_1,B_1)$ where $A_1 \leq 1-2k$ and $B_1 \geq (A_1+k)/(1-k)$ with*

$$k = \frac{2(B-A)(B'-A')}{(B+A+2)(B'+A'+2)+(B-A)(B'-A')}.$$

The result is best possible.

Theorem 4. *If* $f(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$ *belongs to* $\Sigma_p(A, B)$ *and* $g(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} b_n z^n$, *with* $|b_n| \leq 1$, $n \geq 1$, *then* $f(z) * g(z) \in \Sigma(A, B)$.

Proof. Since $f \in \Sigma_p(A, B)$, we have

$$\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} a_n \leq 1.$$

Further $|b_n| \leq 1$, $n \geq 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} |a_n b_n| = \sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} a_n |b_n| \leq 1.$$

This shows that $f(z) * g(z) \in \Sigma(A, B)$. $\Diamond$

Corollary. *If* $f(z) \in \Sigma_p(A, B)$ *and* $g(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} b_n z^n$, $0 \leq b_n \leq 1$ *for* $n \geq 1$ *then* $f(z) * g(z) \in \Sigma_p(A, B)$.

Note that $g(z)$ need not even be univalent.

Remark. Our results generalize the corresponding results for the classes $\Sigma(\alpha)$ and $\Sigma_p(\alpha)$ obtained by Juneja and Reddy [1].

Consider the functions $f(z) = \frac{1}{z} + \frac{B-A}{B+A+2} z$ and $g(z) = \frac{1}{z} + \frac{B-A}{2B+A+3} z^2$ in $\Sigma_p(A, B)$. It is easy to verify that $h(z) = \frac{1}{z} + \frac{B-A}{B+A+2} z + \frac{B-A}{2B+A+3} z^2$ belongs to $\Sigma_p(-1, 1)$ only for values of A and B such that $B - A \leq 3(A+1)^2$ and $h(z)$ need not belong to $\Sigma_p(-1, 1)$ for all A, B. In other words, $f(z), g(z) \in \Sigma_p(A, B)$ need not imply that $h(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} (a_n + b_n) z^n \in \Sigma_p(A_1, B_1)$ for any pair of values A_1, B_1. But we have

Theorem 5. *If* $f(z)$ *and* $g(z)$ *are in* $\Sigma_p(A, B)$, *then* $h(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} (a_n^2 + b_n^2)\, z^n \in \Sigma_p(A_1, B_1)$ *where* $A_1 \leq 1 - 2k$ *and* $B_1 \geq (A_1 + k)/(1-k)$ *with*

$$k = \frac{4(B-A)^2}{(B+A+2)^2 + 2(B-A)^2}.$$

The result is best possible.

Proof. Since $f(z), g(z) \in \Sigma_p(A, B)$,

$$\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} b_n \leq 1.$$

Therefore

$$\sum_{n=1}^{\infty} \left\{ \frac{n(B+1)+A+1}{B-A} a_n \right\}^2 \leq \left[\sum_{n=1}^{\infty} \frac{n(B+1)+A+1}{B-A} a_n \right]^2 \leq 1.$$

Similarly

$$\sum_{n=1}^{\infty} \left\{ \frac{n(B+1)+A+1}{B-A} b_n \right\}^2 \leq 1.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{n(B+1)+A+1}{B-A} b_n \right)^2 (a_n^2 + b_n^2) \leq 1. \tag{11}$$

We want to find values of A_1 and B_1 such that

$$\sum_{n=1}^{\infty} \left(\frac{n(B_1+1)+A_1+1}{B_1-A_1} b_n \right)^2 (a_n^2 + b_n^2) \leq 1. \tag{12}$$

Comparing (12) with (11) we see that (12) is true if

$$\frac{n(B_1+1)+A_1+1}{B_1-A_1} \leq \frac{1}{2} \left(\frac{n(B+1)+A+1}{B-A} \right)^2 = \frac{1}{2} u^2$$

or

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(n+1)}{u^2+2}, \quad n \geq 1. \tag{13}$$

The right hand side of (13) is a decreasing function of n, hence the maximum value is obtained by setting $n=1$. This gives

$$\frac{B_1-A_1}{B_1+1} \geq \frac{4(B-A)^2}{(B+A+2)^2 + 2(B-A)^2} = k. \tag{14}$$

Keeping A_1 fixed in (14) we get $B_1 \geq (A_1+k)/(1-k)$ and $B_1 \leq 1$ gives $A_1 \leq 1-2k$ with k given as in (14).

The functions $f(z) = g(z) = \frac{1}{z} + \frac{B-A}{B+A+2} z$ show that our result is best possible . In fact, $h(z) = \frac{1}{z} + \frac{2(B-A)^2}{(B+A+2)^2} z \in \Sigma_p(1-2k, 1)$ with k as in (14). ◊

Remark. Choosing $A = 2\alpha - 1$ and $B = 1$ in Th. 5 we get the corresponding result of Juneja and Reddy [1].

We have seen in Th. 1 that if $f(z), g(z) \in \Sigma_p(A, B)$, then $f(z) * * g(z) \in \Sigma_p(A_1, B_1)$ where $A_1 \leq 1 - 2k$ and $B_1 \geq (A_1 + k)/(1 - k)$ with $k = 2(B - A)^2/[(B + A + 2)^2 + (B - A)^2]$. Conversely, if $h(z) \in \in \Sigma_p(A_1, B_1)$, A_1, B_1 as described above , do there exist functions $f(z), g(z) \in \Sigma_p(A, B)$ such that $h(z) = f(z) + g(z)$? The answer is negative as is shown by the following example.

Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$, with $f(z), g(z) \in \in \Sigma_p(A, B)$. Then

$$a_n \leq \frac{B - A}{n(B + 1) + A + 1}, \quad b_n \leq \frac{B - A}{n(B + 1) + A + 1}, \quad n \geq 1.$$

and

$$f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n \in \Sigma_p(A_1, B_1)$$

as described in Th. 1. But $a_n b_n \leq (B - A)^2/[n(B + 1) + A + 1]^2$, $n \geq 1$ for the convolution of any two functions in $\Sigma_p(A, B)$. Now consider

$$h(z) = \frac{1}{z} + \frac{B_1 - A_1}{n(B_1 + 1) + A_1 + 1} z^n \in \Sigma_p(A_1, B_1).$$

For $h(z)$ we have

$$\frac{B_1 - A_1}{n(B_1 + 1) + A_1 + 1} \geq \frac{(B - A)^2}{(n(B + 1) + A + 1)^2}$$

for $n \geq 1$ as in Th. 1. This shows that there is no $f(z)$ and $g(z)$ in $\in \Sigma_p(A, B)$ for which $h(z) = f(z) * g(z)$, though $h(z) \in \Sigma_p(A_1, B_1)$.

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