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RADICALS WHICH COINCIDE ON A CLASS OF RINGS

A. D. **Sands**

Department of Mathematics, Dundee University, Dundee DD1 4HN, Scotland

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Abstract: In this paper the coincidence of radical classes on a given class of rings is considered in three senses. It is asked whether for a given radical and given class of rings there must be smallest or largest radicals coinciding with the given radical. For each case it is either shown to be so or a counter-example is provided. Additional conditions are introduced which guarantee such existence in these cases.

We shall work in the class of associative rings but shall not assume that each ring has an identity element. The fundamental definitions and results on radical theory may be found in Divinsky [2] and Wiegandt [5].

If $\mathbf{C}$ is any class of rings there is a smallest radical class containing $\mathbf{C}$ which we shall denote by $l(\mathbf{C})$. There is also a smallest semisimple class containing $\mathbf{C}$ and, corresponding to it, there is a largest radical class such that all rings in $\mathbf{C}$ are semisimple. We shall denote this radical class by $u(\mathbf{C})$. Equivalently

$$u(\mathbf{C}) = \{R : \text{no non-zero homomorphic image of } R \text{ is isomorphic to an accessible subring of a ring in } \mathbf{C}\}.$$

As usual we assume that classes of rings are closed with respect to the formation of isomorphic images.

Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be radicals and let $\mathbf{C}$ be a class of rings. Divinsky [1], [2] and Mlitz [3] have considered the coincidence of $\mathbf{R}_1$ and $\mathbf{R}_2$ on $\mathbf{C}$ in various senses. Mlitz works in the wider class of $\mathbf{\Omega}$-groups and consider both radicals in our sense, i.e. Kurosh-Amitsur radicals, and also in the wider sense of Hoehnke radicals. So while certain of his results hold, usually with the same proof, in the present setting, this is not so for all his results and we shall give some examples to illustrate this. The coincidence of radicals $\mathbf{R}_1$ and $\mathbf{R}_2$ on $\mathbf{C}$ is defined in three senses:

1. For each A in $\mathbf{C}$, A is $\mathbf{R}_1$-radical if and only if A is $\mathbf{R}_2$-radical.
2. For each A in $\mathbf{C}$, A is $\mathbf{R}_1$-semisimple if and only if A is $\mathbf{R}_2$-semisimple.
3. For each A in $\mathbf{C}$, $\mathbf{R}_1(A) = \mathbf{R}_2(A)$.

Divinsky [1], [2] refers to (1) as coincidence in the weak sense and to (3) as coincidence in the strong sense. Mlitz [3] refers to (1) as r-coincidence, to (2) as s-coincidence and to (3) as coincidence. We shall use this notation of Mlitz.

Mlitz points out that if $\mathbf{C}$ is homomorphically closed then s-coincidenc e implies coincidence, while if $\mathbf{C}$ is hereditary, i.e. closed under formation of ideals, r-coincidence implies coincidence.

If $\mathbf{R}_1$ and $\mathbf{R}_2$ are radicals coinciding on a class $\mathbf{C}$ in any of these senses and $\mathbf{R}_3$ is a radical lying between $\mathbf{R}_1$ and $\mathbf{R}_2$ then it also coincides on $\mathbf{C}$ with them in the same sense. So it is natural to consider whether there is a smallest or greatest radical coinciding with a given radical $\mathbf{R}$ on a class $\mathbf{C}$ in each sense. Mlitz has shown that, for a given radical $\mathbf{R}$, there is a smallest radical r-coincident with $\mathbf{R}$ on $\mathbf{C}$. It is obtained by taking the intersection of all such radical classes. However no largest such radical need exist. There is a largest such radical s-coincident with $\mathbf{R}$ on $\mathbf{C}$. It is obtained as the upper radical of the class which is the intersection of the semisimple classes of all such radicals. However no smallest such radical need exist. For radicals coinciding with $\mathbf{R}$ on $\mathbf{C}$ the corresponding constructions yield both a smallest and a largest such radical. However these need not equal, respectively, the smallest r-coincident radical nor the largest s-coincident radical.

We illustrate these situations with the following examples. Let E, F and G be distinct, i.e. non-isomorphic, fields.

Example 1. Let $\mathbf{C} = \{0, E \oplus F\}$. Let $\mathbf{R}_1$ be any radical such that E is radical and F is semisimple. Let $\mathbf{R}_2$ be any radical such that

E is semisimple and F is radical. Then $\mathbf{R}_1$, $\mathbf{R}_2$ are r-coincident and s-coincident on $\mathbf{C}$, but are not coincident. So, in general, r- and s-coincidence together do not imply coincidence. The smallest radical r-coinciding with $\mathbf{R}_1$ on $\mathbf{C}$ is the zero radical, where only 0 is radical. If a largest such radical existed it would contain $\mathbf{R}_2$ and also, of course, $\mathbf{R}_1$. This is not possible as then $E \oplus F$ would belong to this radical and to $\mathbf{C}$ but not to $\mathbf{R}_1$. The largest radical s-coinciding with $\mathbf{R}_1$ on $\mathbf{C}$ is the radical consisting of all rings. If a smallest such radical existed it would be contained in $\mathbf{R}_1$ and also $\mathbf{R}_2$. However $E \oplus F$ is semisimple for such a radical but not for $\mathbf{R}_1$. So no smallest such radical can exist. Any radical coinciding with $\mathbf{R}_1$ must contain E and so $l\{E\}$ is the smallest such radical, which is strictly larger than the smallest r-coincident radical. Similarly F must be in the semisimple class of any radical coinciding with $\mathbf{R}_1$ on $\mathbf{C}$. So the largest such radical is $u\{F\}$ which is strictly smaller than the largest s-coincident radical.

Example 2. Let $\mathbf{C} = \{O, E, E \oplus F, E \oplus F \oplus G\}$. Let σ be a radical such that E is radical and F, G are semisimple. Then it is routine to check that the largest radical s-coincident with σ on $\mathbf{C}$ is the class of all rings, the largest radical r-coincident with σ on $\mathbf{C}$ is $u(F)$ and the largest radical coinciding with σ on $\mathbf{C}$ is $u(F, G)$. These three largest radicals are distinct. Furthermore, since the upper radicals involved are generated by special classes of rings, these radicals are also the greatest such Hoehnke radicals. So we have an example which provides an answer to Problem 1 of [3].

The next examples are intended to emphasise the differences between the cases of Hoehnke radicals studied in [3] and the Kurosh-Amitsur radicals used here. We show that certain relationships between these coincidences given there do not hold in the present setting.

Example 3. Let $\mathbf{C} = \{O, E, F, E \oplus F, E \oplus F \oplus G\}$. Let σ be a radical such that E, G are radical and F is semisimple . Then it is routine to check that $l(E)$ is the smallest radical r-coinciding and s-coinciding with σ on C, but that it does not coincide with it there. This example shows that Th. 1 of [3], which holds for Hoehnke radicals, does not hold for Kurosh-Amitsur radicals, i.e. $\rho_r = \rho_s$ does not imply $\rho_r = \rho$.

Example 4. Let $\mathbf{C}$ be as in example 3 and let σ be as in example 2. Then $l(E)$ is the smallest radical coinciding with σ on $\mathbf{C}$ in all three senses. The largest radical s-coinciding with σ on $\mathbf{C}$ is $u(F)$, while the largest radical coinciding with σ on $\mathbf{C}$ is $u(F, G)$. So, for reasons similar to those given in Example 3, Th. 2 of [3] does not hold for Kurosh-Amitsur radicals.

The results on these smallest and largest radical classes have been presented in an abstract manner in terms of intersections of radical classes or semisimple classes. We now present them in terms of classes of rings associated with the radical class $\mathbf{R}$ and the class $\mathbf{C}$. We shall use the following notations:

$\mathbf{C}_1 = \{R \in \mathbf{C} : R \text{ is } \mathbf{R}\text{-radical}\}$;
$\mathbf{C}_2 = \{R \in \mathbf{C} : R \text{ is } \mathbf{R}\text{-semisimple}\}$;
$\mathbf{C}_3 = \{R : \mathbf{R}(A) = R \text{ for some } A \in \mathbf{C}\}$;
$\mathbf{C}_4 = \{R : A/\mathbf{R}(A) = R \text{ for some } A \in \mathbf{C}\}$.

We note that $\mathbf{C}_1$ and $\mathbf{C}_2$ are subclasses of $\mathbf{C}$ but that $\mathbf{C}_3$ and $\mathbf{C}_4$ need not be so. $\mathbf{C}_1$ is a subclass of $\mathbf{C}_3$ and equality occurs if $\mathbf{C}$ is closed under formation of ideals. $\mathbf{C}_2$ is a subclass of $\mathbf{C}_4$ and equality occurs if $\mathbf{C}$ is closed under formation of homomorphic images.

Theorem 1. *Let* $\mathbf{R}$ *be a radical and let* $\mathbf{C}$ *be a class of rings. Then* $l(\mathbf{C}_1)$ *is the smallest radical* r*-coinciding with* $\mathbf{R}$ *on* $\mathbf{C}$.

Proof. Let α be any radical r-coinciding with $\mathbf{R}$ on $\mathbf{C}$. Then A in $\mathbf{C}_1$ implies that A is in $\mathbf{C}$ and that A is $\mathbf{R}$-radical. Therefore A is α-radical. Hence $l(\mathbf{C}_1) \subseteq \alpha$. In particular we have that $l(\mathbf{C}_1) \subseteq \mathbf{R}$ and so if A is $l(\mathbf{C}_1)$-radical it is $\mathbf{R}$-radical. Conversely if A is in $\mathbf{C}$ and is $\mathbf{R}$-radical then A is in $\mathbf{C}_1$ and so is $l(\mathbf{C}_1)$-radical. $\Diamond$

Theorem 2. *Let* $\mathbf{R}$ *be a radical and let* $\mathbf{C}$ *be a class of rings. Then* $u(\mathbf{C}_2)$ *is the largest radical* s*-coinciding with* $\mathbf{R}$ *on* $\mathbf{C}$.

Proof. Let α be any radical s-coinciding with $\mathbf{R}$ on $\mathbf{C}$. If A is in $\mathbf{C}_2$ then A is in $\mathbf{C}$ and is $\mathbf{R}$-semisimple. Therefore it is α-semisimple. It follows that $\alpha \subseteq u(\mathbf{C}_2)$. In particular we have $\mathbf{R} \subseteq u(\mathbf{C}_2)$ and so if A is in $\mathbf{C}$ and is $u(\mathbf{C}_2)$-semisimple it is $\mathbf{R}$-semisimple. Conversely if A is in $\mathbf{C}$ and is $\mathbf{R}$-semisimple then A is in $\mathbf{C}_2$ and so is $u(\mathbf{C}_2)$-semisimple. $\Diamond$

Theorem 3. *Let* $\mathbf{R}$ *be a radical and let* $\mathbf{C}$ *be a class of rings. Then* $l(\mathbf{C}_3)$ *is the smallest radical coinciding with* $\mathbf{R}$ *on* $\mathbf{C}$.

Proof. Let α be any radical coinciding with $\mathbf{R}$ on $\mathbf{C}$. If A is in $\mathbf{C}_3$ then there exists R in $\mathbf{C}$ such that $A = \mathbf{R}(R) = \alpha(R)$. Hence A is α-radical and so $l(\mathbf{C}_3) \subseteq \alpha$. In particular $l(\mathbf{C}_3)(R) \subseteq \mathbf{R}(R)$. Conversely for each R in $\mathbf{C}$ we have $\mathbf{R}(R)$ in $\mathbf{C}_3$ and so $\mathbf{R}(R) \subseteq l(\mathbf{C}_3)(R)$. $\Diamond$

Theorem 4. *Let* $\mathbf{R}$ *be a radical and let* $\mathbf{C}$ *be a class of rings. Then* $u(\mathbf{C}_4)$ *is the largest radical coinciding with* $\mathbf{R}$ *on* $\mathbf{C}$.

Proof. Let α be any radical coinciding with $\mathbf{R}$ on $\mathbf{C}$. If A is in $\mathbf{C}_4$ then there exists R in $\mathbf{C}$ with $A = R/\mathbf{R}(R) = R/\alpha(R)$. It follows that

$\alpha \subseteq u(\mathbf{C}_4)$. In particular we have $\mathbf{R}(R) \subseteq u(\mathbf{C}_4)(R)$. Conversely for each R in $\mathbf{C}$ the ring $R/\mathbf{R}(R)$ is in $\mathbf{C}_4$ and so is $u(\mathbf{C}_4)$-semisimple. Therefore $u(\mathbf{C}_4)(R) \subseteq \mathbf{R}(R)$. ◊

If $\mathbf{C}$ is closed under formation of ideals then $\mathbf{C}_1 = \mathbf{C}_3$ and so $l(\mathbf{C}_1) = l(\mathbf{C}_3)$. Dually if $\mathbf{C}$ is closed under formation of homomorphic images then $\mathbf{C}_2 = \mathbf{C}_4$ and so $u(\mathbf{C}_2) = u(\mathbf{C}_4)$. However in these cases the stronger results hold that in the first case r-coincidence implies coincidence and in the second case s-coincidence implies coincidence. The proofs by Mlitz [3] for these results still hold in this case. Under the first of these conditions a largest radical r-coinciding with $\mathbf{R}$ on $\mathbf{C}$ exists, being equal to the largest coincident radical . Dually under the second of these conditions a smallest radical s-coinciding with $\mathbf{R}$ on $\mathbf{C}$ exists, being equal to the smallest coincident radical . If both conditions hold then all three senses of coincidence are equivalent. In particular this is true for the class of all associative rings. Also if both conditions hold then all three smallest radicals are equal as are all three largest radicals. However such equalities hold under somewhat weaker c onditions.

Theorem 6. *Let $\mathbf{C}$ be a class of rings closed under the formation of ideals and let $\mathbf{R}$ be a radical. If $\mathbf{C}_1$ is closed under formation of homomorphic images then $l(\mathbf{C}_1)$ is the smallest radical coinciding with $\mathbf{R}$ on $\mathbf{C}$ in each of the three senses.*

Proof. By Theorems 1, 3 and Prop. 1, [3], this result holds for senses (1) and (3). Then, since $l(\mathbf{C}_1)$ coincides with $\mathbf{R}$ on $\mathbf{C}$ it s-coincides there also . Let α be any radical s-coinciding with $\mathbf{R}$ on $\mathbf{C}$. Let A be in $\mathbf{C}_1$. Then, since $\mathbf{C}_1$ is closed under formation of homomrphic images, $A/\alpha(A)$ is in $\mathbf{C}_1 \subseteq \mathbf{C}$. As $A/\alpha(A)$ is α-semisimple it is also $\mathbf{R}$-semisimple. However, since A is in $\mathbf{C}_1$, we have that $A/\alpha(A)$ is $\mathbf{R}$-radical. Therefore $\alpha(A) = A$ and so $l(\mathbf{C}_1) \subseteq \alpha$. ◊

Theorem 6. *Let $\mathbf{C}$ be a class of rings closed under formation of homomorphic images and let $\mathbf{R}$ be a radical. If $\mathbf{C}_2$ is closed under formation of ideals then $u(\mathbf{C}_2)$ is the largest radical coinciding with $\mathbf{R}$ on $\mathbf{C}$ in each of the three senses.*

Proof. By Theorems 2, 4 and Prop. 1, [3], this result holds for senses (2) and (3). Since $u(\mathbf{C}_2)$ coincides with $\mathbf{R}$ on $\mathbf{C}$ it r-coincides also. Let α be any radical r-coinciding with $\mathbf{R}$ on $\mathbf{C}$. Let A be in $\mathbf{C}_2$. Since $\mathbf{C}_2$ is closed under formation of ideals we have that $\alpha(A)$ is in $\mathbf{C}_2$ and so is in $\mathbf{C}$. It follows that $\alpha(A)$ is $\mathbf{R}$-radical. However $\alpha(A)$ is in $\mathbf{C}_2$ and so is $\mathbf{R}$-semisimple. Therefore $\alpha(A) = O$ and so $\mathbf{C}_2 \subseteq S_\alpha$, the semisimple class of α. It follows that $\alpha \subseteq u(\mathbf{C}_2)$. $\Diamond$

We now turn to some important examples of classes of rings and of radicals where these results may be applied. Divinsky [1], [2] considered the case where $\mathbf{A}$ is the class of artinian rings and the classical radical is used, i.e., for each $R \in \mathbf{A}$ the radical of R is the maximal nilpotent ideal $N(R)$. Many of the radicals first considered were generalisations of this radical to the class of all associative rings, defined so as to coincide with it on the class of artinian rings. $\mathbf{A}$ is closed under the formation of homomorphic images but is not closed under the formation of ideals. However the semisimple class is closed under the formation of ideals. So Th. 6 does apply in this case. By means of examples Divinsky showed that the smallest radical class which is r-coincident in this case is strictly contained in the smallest coincident radical class, which, by [3], is also the smallest s-coincident radical class. He showed by means of an example that this smallest coincident radical class is strictly contained in the lower Baer radical, i.e. the radical class generated by all nilpotent rings. He also showed that the largest radicals in the weak and the strong sense are equal. By Th. 6 this is also the the largest s-coincident radical class.

In [4] the class of artinian rings is considered also, but using the von Neumann radical $\mathbf{Q}$ or the hereditarily idempotent radical $\mathbf{H}$. $\mathbf{Q}$ is the class of all rings R such that for each $a \in R$ the equation $axa = a$ has a solution in R. $\mathbf{H}$ is the class of all rings R such that every ideal A of R is idempotent. The radical $\mathbf{Q}$ is strictly contained in $\mathbf{H}$, but they coincide on the class $\mathbf{A}$ of artinian rings. The $\mathbf{H}$-radical rings in $\mathbf{A}$ are precisely the 'semisimple' rings in the classical sense. The $\mathbf{H}$-semisimple rings in $\mathbf{A}$ are the rings R in $\mathbf{A}$ such that the maximal nilpotent ideal $N(R)$ is essential in R. It is shown in [4] that the smallest radicals in all three senses are equal. Further details of this case are given there.

It is natural also to consider the class $\mathbf{N}$ of noetherian rings. This class is also closed under the formation of homomorphic images but not under the formation of ideals. The nil radical (upper Baer radical) of a noetherian ring is nilpotent. The trivial ring defined on the infinite cyclic group Z by making all products equal to 0 is noetherian. It generates as its lower radical the lower Baer radical β. Let $\mathbf{R}$ be any radical lying between the lower and upper Baer radicals, inclusive. Then $\mathbf{R}(R)$ is nilpotent for each $R \in N$. It follows that the smallest radical in all three senses coinciding with $\mathbf{R}$ on $\mathbf{N}$ is the lower Baer radical.

We note that this result does not extend to the Jacobson Radical. For example the ring of p-adic integers is noetherian and has nil radical equal to O but has non-zero Jacobson radical.

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