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SUBVARIETIES OF SEGRE VARIETIES AND TRISECANT LINES

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Abstract: We study surfaces, X, embedded in the Segre variety $\mathbf{P} \times \mathbf{P}^s \subset \subset \mathbf{P}^{2s+1}$ which do not have trisecant lines. If $s = 3$ and X is smooth we prove that $\deg(X) \leq 37$.

Let $U(r, s) \equiv \mathbf{P}^r \times \mathbf{P}^s \subset \mathbf{P}^{rs+r+s}$ be the Segre variety defined over an algebraically closed field $\mathbf{K}$. Our main interest is when $\mathbf{K}$ is the algebraic closure of a finite field $GF(q)$, but even if $\mathrm{char}(\mathbf{K}) = 0$ to the best of our knowledge the key results were previously unknown. Here we study surfaces $X \subset U(1, s)$ without trisecant lines (in the sense of Def. 1.1). Denote by $PG(N, q)$ the projective space of dimension N defined over $GF(q)$. There is a close relation between such surfaces and objects coming from Galois geometries, namely K-caps, $K \in \mathbf{N}$,

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in $PG(N,q)$; by definition a K-cap A of $PG(N,q)$ is a set of K points in $PG(N,q)$ such that no 3 points of A are collinear. However, when we claim to have proved the existence of at least a trisecant line D on a surface Y defined over $GF(q)$, it could happen that D is not defined over $GF(q)$ but only over some finite (a priori unknown) extension of $GF(q)$; even if D is defined over $GF(q)$, it could be that $D \cap GF(q)$ does not contain points defined over $GF(q)$. If $r = 1$ and s is low, the non existence of trisecant lines is a very strong restriction on X and such surfaces are very few (see Th. 0.1 below for a precise statement).

Theorem 0.1. *Assume* $\text{char}(K) \neq 2,3$. *Let* $X \subset U(1,3)$ *be a smooth surface without trisecant lines. Then* $\deg(X) \leq 37$.

Th. 0.1 will be proved in Section 3. We do not know what is the situation for low $\deg(X)$. We expect that no such example exists, unless $\deg(X)$ is very, very small. For large s, say $s \geq 6$, the non existence of trisecant lines is a very mild restriction (see 2.7 and Remark 1.4).

1. Preliminary remarks

We work over an algebraically closed field $\mathbf{K}$. Our main interest is when $\mathbf{K}$ is the algebraic closure of a finite field $GF(q)$. Fix integers $r \geq 1$, $s \geq 1$. Let $U(r,s) \simeq \mathbf{P}^r \times \mathbf{P}^s \subset \mathbf{P}^{rs+r+s}$ be the Segre variety. Let $\pi' : U(r,s) \to \mathbf{P}^r$ and $\pi" : U(r,s) \to \mathbf{P}^s$ be the projections. Set $O_{U(r,s)}(1,0) := \pi'^*(O_{\mathbf{P}^r}(1))$ and $O_{U(r,s)}(0,1) := \pi"^*(O_{\mathbf{P}^s(1)})$. By definition of Segre embedding we have $O_{U(r,s)}(1) = O_{U(r,s)}(1,0) + O_{U(r,s)}(0,1)$. Hence we have $\deg(U(r,s)) = (r+s)!/r!s!$.

Definition 1.1. Let $W\mathbf{P}^N$ be a closed subscheme (even not reduced or not irreducible). A line $L \subset \mathbf{P}^N$ will be called a *trisecant line* of W if the scheme $W \cap L$ contains a length 3 subscheme of L, i.e. if and only if either L is contained in W or $W \cap L$ is finite but contains at least 3 points or $\text{card}(W \cap L) = 1$ or 2, but the sum of the multiplicities of the divisor $W \cap L$ of the smooth curve L at the points of $W \cap L$ (i.e. its degree) is at least 3.

Remark 1.2. Let $W \subset U(r,s)$ be a closed integral variety. Assume that W has no trisecant line. Set $w := \dim(W)$ and assume $r \leq w \leq s$. Assume $\pi'(W) = \mathbf{P}^r$ and $\dim(\pi"(W)) = w$. For general $P \in \mathbf{P}^r$ and $Q \in \pi"(W)$, let a be the degree of the subscheme ${\pi'}^{-1}(P) \cap W$ of $\mathbf{P}^s$ and b the degree of the 0-dimensional subscheme ${\pi"}^{-1}(Q) \cap W$ of $\mathbf{P}^r$, i.e. set $b := \deg(\pi"|W)$. Since W has no trisecant line, if $r = 1$ we have

$1 \le b \le 2$. We have $\deg(W) := \sum_{0 \le j \le r}(w!/(w-j)!j!)O_{U(r,s)}(1,0)j \cdot \cdot O_{U(r,s)}(0,1)^{w-j} \cdot W$. We have $a = O_{U(r,s)}(1,0)^r \cdot O_{U(r,s)}(0,1)^{w-r} \cdot W$ and $b \cdot \deg(\pi"(W)) = O_{U(r,s)}(0,1)^w \cdot W$. Note that for a general linear subspace M of $\mathbf{P}^r$, we have $\deg(\pi"|(\pi'^{-1}(M) \cap W)) \le b$. Hence if M is a general hyperplane of $\mathbf{P}^r$ the subscheme $\pi'^{-1}(M) \cap W$ of dimension $w-1$ of $M \times \mathbf{P}^s \simeq U(r-1,s)$ has invariants a' and b' with $a' = a$ and $b' \le b$. Hence by induction on r and w we obtain the existence of an upper bound for the integer $\deg(W)$ depending only on w, r, s, a, b and on the numerical invariants of $\pi'^{-1}(M) \cap W$. We were unable to obtain an explicit closed form for such upper bound. If $r = 1$ we have $\deg(W) = b \cdot \deg(\pi"(W)) + wa$.

Remark 1.3. Since $O_{U(r,s)}(1) = O_{U(r,s)}(1,0) + O_{U(r,s)}(0,1)$ and both $O_{U(r,s)}(1,0)$ and $O_{U(r,s)}(0,1)$ are spanned, we see that every line $D \subset \subset U(r,s)$ is contained in one of the rulings of $U(r,s)$, i.e. either $\pi'(D)$ is a point or $\pi"(D)$ is a point.

Remark 1.4. Fix integers w, r with $w > r > 0$. Let W be an irreducible projective variety with $\dim(W) = w$. It is a strong condition on the biregular type of W the existence of a nonconstant morphism $f : W \to \mathbf{P}^r$. Hence if $W \subset U(1,s)$ and $w \ge 2$, it is a strong condition on the biregular type of W the fact that W is not contained in a fiber of π'.

Remark 1.5. Let X be a closed subscheme of $U(r,s)$. Since the homogeneous ideal of $U(r,s)$ in $\mathbf{P}^{rs+r+s}$ is generated by quadrics, every trisecant line to X is contained in $U(r,s)$ by Bézout theorem. Hence by Remark 1.3 every trisecant line to X is contained in one of the rulings of $U(r,s)$.

Remark 1.6. Let $W \subset U(r,s)$ be an integral variety. Assume that W has no trisecant lines. Set $w := \dim(W)$ and assume $w \le r \le s$. Assume $\dim(\pi'(W)) = \dim(\pi"(W)) = w$. For general $P \in \pi'(W)$ and $Q \in \pi"(W)$, let a be the degree of the subscheme $\pi'^{-1}(P) \cap W$ of $\mathbf{P}^s$ and b the degree of the subscheme $\pi"^{-1}(Q)$ of $\mathbf{P}^r$. We have $a \cdot \deg(\pi'(W)) = = O_{U(r,s)}(1,0)^w \cdot W$ and $b \cdot \deg(\pi"(W)) = O_{U(r,s)}(0,1)^w \cdot W$. We have $\deg(W) := \sum_{0 \le j \le w}(w!/(w-j)!j!)O_{U(r,s)}(1,0)^j \cdot O_{U(r,s)}(0,1)^{w-j} \cdot W$. Note that for a general linear subspace M of $\mathbf{P}^r$, we have $\deg(\pi"|(\pi'^{-1}(M) \cap W)) \le b$. Hence if M is a hyperplane the subscheme $\pi'^{-1}(M) \cap W$ of dimension $w-1$ of $M \times \mathbf{P}^s \simeq U(r-1,s)$ has invariants a' and b' with $a' = a$ and $b' \le b$. The same remark works for general linear subspaces of $\mathbf{P}^s$. Hence by induction on w, r and s

we obtain the existence of an upper bound for the integer $\deg(W)$ as a function of the integers w, r, s, a, b and of the numerical invariants of general linear sections of W in $U(r-1,s)$ and $U(r,s-1)$. As in 1.2 we were unable to obtain an explicit closed form for such upper bound.
Remark 1.7. Let $T \subset \mathbf{P}^3$ be an integral curve without trisecant lines. Hence T is not a line. If T is a plane curve, then $\deg(T) = 2$ by Bézout, i.e. T is a smooth conic. Assume that T spans $\mathbf{P}^3$. Fix a general $P \in T$ and consider the projection $T' \subset \mathbf{P}^2$ of T from P. If $T \to T'$ is not biregular, then there is a trisecant line to T containing P, contradiction. Hence we may assume that $T \to T'$ is biregular. Since $T \to T'$ is birational we have $\deg(T') = \deg(T) - 1$. By Castelnuovo's upper bound for the arithmetic genus of a space curve ([8, Th. IV.6.4]) and the arithmetic genus of a plane curve we see that either T is a rational normal curve or T is a quartic curve with $p_a(T) = 1$. In the latter case T is either a smooth elliptic curve or a rational curve with a double point which is either an ordinary node or an ordinary cusp.

2. Surfaces in U(1,s)

In this section and the next one we will consider the case $r = 1$ and $w = 2$. Let $X \subset U(1,s)$ be an integral projective surface without trisecant lines. Furthermore, we assume that X is not contained in a fiber of the first projection π'. Let a be the degree of the curve ${\pi'}^{-1}(P) \cap X \subset \mathbf{P}^s$. This degree does not depend on the choice of $P \in \mathbf{P}^1$ if we count the multiplicities of the components of the scheme ${\pi'}^{-1}(P) \cap X$. Since the fibers of $\pi"$ are lines and by assumption X contains no line, $\pi"|X$ is finite. In particular $\dim(\pi"(X)) = 2$. Furthermore, since X has no trisecant line, for every $Q \in \pi"(X)$ the scheme ${\pi"}^{-1}(Q) \cap W$ has length ≤ 2, i.e. it is an effective divisor of the line ${\pi"}^{-1}(Q)$ with degree 1 or 2. By [7, Prop. 10.2], this degree is the same for all $Q \in \pi"(X)_{\text{reg}}$. This integer, b, is the degree of the map $\pi"|X$. Hence either $b = 1$ or $b = 2$. By Remark 1.2 $\deg(X) = b \cdot \deg(\pi"(X)) + 2a$.
Remark 2.1. Since X has no trisecant line, $X \neq U(1,1)$, i.e. the case $s = 1$ is impossible.
Remark 2.2. Assume $s = 2$. Hence X is a divisor of $U(1,2)$. By the definitions of the integers a and b, X is a divisor of $O_{U(1,2)}(b,a)$. Since the curves ${\pi'}^{-1}(P) \cap X$, $P \in \mathbf{P}^1$, are plane curves without trisecant lines, every curve ${\pi'}^{-1}(P) \cap X$ is a smooth conic. In particular $a =$

$= 2$. We have $b = 1$ or 2. Viceversa, every integral divisor of type $(1, 2)$ or of type $(2, 2)$ such that all the corresponding conics are smooth has no trisecant line. However, we will show that there is no such X without trisecant lines. The family $\mathbf{S} := \{\pi''(\pi'^{-1}(P) \cap X)\}_{P \in \mathbf{P}^1}$ is a family of smooth plane conics parametrized by a complete curve, i.e. $\mathbf{P}^1$. However, the variety U of smooth plane conics is an affine variety because it is an open set of the space $\mathbf{P}^5$ parametrizing all conics and $\mathbf{P}^5 \setminus U$ is a hypersurface $\Delta \neq \emptyset$ (the zero-locus of the discriminant). Hence the map $\mathbf{S} \to U$ is constant, i.e. $\pi''(X)$ is a conic, i.e. all fibers of $\pi''|X$ are lines. Hence X contains infinitely many lines, contradiction.

(2.3) Here we assume $s = 3$. Since the curves $C'(P) := \pi'^{-1}(P) \cap$ $\cap X$, $P \in \mathbf{P}^1$, are space curves without trisecant lines, by Remark 1.7 we have $2 \leq a \leq 4$. Furthermore, if $a = 2$ or 3, then each $C'(P)$ is a smooth rational curve, while if $a = 4$, then we have $p_a(C'(P)) = 1$ for every P. In the next two remarks we will exclude the case $a = 3$ and the case $a = 2$.

Remark 2.4. Here we exclude the case $a = 3$. Let $X \subset U(1, 3)$ be an integral surface without trisecant lines and with $a = 3$. First note that every degree 3 curve $C'(P) := \pi'^{-1}(P) \cap X$ is a smooth rational normal curve because it cannot be a plane curve (otherwise it would have trisecant lines) and cannot have a line as an irreducible component. Thus we find a family $\{C'(P)\}_{P \in \mathbf{P}^1}$ of smooth rational normal curves in $\mathbf{P}^3$ parametrized by a complete curve, $\mathbf{P}^1$. Such a family must be constant because the variety, Z, of all rational normal curves is affine (e.g. use that $Z \simeq PGL(4)/PGL(2)$ because it is parametrized by a choice of a base of $H^0(\mathbf{P}^1, O_{\mathbf{P}^1}(3))$ modulo $\mathrm{Aut}(\mathbf{P}^1)$ and $PGL(4)/PGL(2)$ is affine because both $PGL(4)$ and $PGL(2)$ are reductive groups). Hence the projection $\pi''|X$ sends each $C'(P)$ to the same curve, i.e. it has lines as fibers, contradiction.

Remark 2.5. Since $PGL(s+1)/PGL(2)$ is an affine variety, the proof of 2.4 shows easily that the case $s = a \geq 2$ and with all fiber smooth rational normal curves is impossible. If $s = 2$ this is a priori the only possible case, because every plane curve (reducible or irreducible) which is not a smooth conic has a trisecant line, either by Bézout (if its degree is > 2) or as an irreducible component. Hence we have checked again Remark 2.2.

(2.6) Here we assume $s = 4$ and that X has at most isolated singularities. Consider again the curves $C'(P) := \pi'^{-1}(P) \cap X$, $P \in$

$\in \mathbf{P}^1$. Set again $a := \deg(C'(P))$ and $g := p_a(C'(P))$. If some $C'(P)$ does not span $\mathbf{P}^4$, we may apply Remark 1.7 and find that either all $C'(P)$ are smooth plane conics or $(a, g) = (3, 0)$ or $(4, 1)$ (and in the first case all curves are smooth).

(2.6.1) Here we assume that for general P the curve $C'(P)$ is smooth. Since the curve $C'(P)$ has no trisecant line we have $(a-2)(a-3)(a-4)/6 = g(a-4)$ (see [6], [12], [13], [15] and the discussion in [4, §4], of why this is true even in positive characteristic). Hence, as in [4, Step 2 of the proof of Th. 4.1], we obtain that if $C'(P)$ spans $\mathbf{P}^4$ only the following pairs of integers (a, g) may occur: $(8, 5)$, $(6, 2)$, $(5, 1)$, or $(4, 0)$.

(2.6.2) Here we assume that for general P the curve $C'(P)$ is singular. We will find that $\mathrm{char}(\mathbf{K}) = 2$. Since X has isolated singularities, if $\mathrm{char}(K) = 0$ then $C'(P)$ is smooth by the generic smoothness theorem ([8, p. 272]). Hence we may assume $p := \mathrm{char}(\mathbf{K}) > 0$. This type of fibrations were first considered in [17]. For a detailed study in the case $p = 2$ or 3, see [5]. Since $C'(P)$ must have at least a cusp of weight $\geq (p-1)/2$ ([17]), say at some $o \in C'(P)$ with $o \in X_{\mathrm{reg}}$, the tangent line D to $C'(P)$ at o has intersection multiplicity $> (p-1)/2$ at o. Since X has no trisecant line in the sense of Def. 1.1, we have $p \leq 3$. The case $p = 3$ is analyzed in [5, part (b) of Prop. 1, Prop. 2 and Fig. 1].

(2.7) Here we discuss the case $w = 2$, $r = 1$ and $s \geq 5$. First we recall the following fact. Let $D \subset \mathbf{P}^N$ be an integral nondegenerate curve. For every integer $t \geq 1$, let $S^t(D) \subseteq \mathbf{P}^N$ be the t-secant variety of D, i.e. the closure of the union of the linear spaces spanned by t general points of D. By [1, Cor. 1.5], we have $\dim(S^t(D)) = \min\{N, 2t-1\}$ for every $t \geq 1$. We fix an integral projective surface X such that there is a surjective morphism $f : X \to \mathbf{P}^1$. We assume that every singular point of X has embedding dimension $\leq s$ and that there are at most finitely many singular points of X with embedding dimension $\geq s-1$. We fix f and we will show that if $s \geq 6$ there are several embeddings, i, of X into $U(1, s)$ such that $f = \pi'|i(X)$ and $i(X)$ has no trisecant line. We fix any embedding $j : X \to \mathbf{P}^m$, m large, $m \geq s \geq 6$ such that $j(X)$ has no trisecant line; for instance, given any embedding of X into $\mathbf{P}^5$ we may take as j the composition of this embedding with a Veronese embedding of order ≥ 2 of $\mathbf{P}^5$. Note that the dimension of the union Prod of the 3-secant varieties $f^{-1}(P)$, $P \in \mathbf{P}^1$, is at most 6. Hence if $m \geq 7$, a general projection of $j(X)$ into $\mathbf{P}^{m-1}$ (i.e. a projection of $j(X)$ from a point of $\mathbf{P}^m \setminus \prod$ such that the corresponding morphism is an embedding of X) will be an embedding, u, of X into $\mathbf{P}^{m-1}$ such that for every $P \in \mathbf{P}^1$ the curve $u(f^{-1}(P))$ has no trisecant line. Then if $m - 1 > s$ we make the same construction $m - 1 - s$ times and we find an embedding u' of X into $\mathbf{P}^s$ such that for every $P \in \mathbf{P}^1$ the curve $u'(f^{-1}(P))$ has no trisecant line. Thus by construction the map $v := (f, u') : X \to U(1, s)$ is an embedding such that $v(X)$ has no trisecant line. The same construction shows that if $s = 5$ we find an embedding $v := (f, u') : X \to U(1, 5) \subset \mathbf{P}^{11}$ such that $v(X)$ has only a finite number $\neq 0$ of trisecant lines.

3. Proof of Theorem 0.1

In this section we will prove Th. 0.1. Fix a smooth surface $X \subset \subset U(1, 3)$ without trisecant lines. Let $\pi : X \to \mathbf{P}^3$ the composition of the inclusion of X in $U(1, 3)$ and the projection $\pi" : U(1, 3) \to \mathbf{P}^3$. Since the fibers of the projection $\pi"$ are embedded as lines in the Segre embeddings, every fiber of π must be a finite subscheme with length ≤ 2. In particular, π is finite and either π is birational or $\deg(\pi) = 2$.

(3.1) Here we assume that π is birational. Set $S := \pi(X)$. Since X has no trisecant line, we see that S has no point of multiplicity ≥ 3. Set $m := \deg(S)$. Let n be the number of pinch points of S and let $z \geq 0$ be the degree of the one dimensional part, D, of $\mathrm{Sing}(S)$. Since S has no triple point, either $D = \emptyset$, i.e. $z = 0$, or D is formed by double points. A priori S may have also isolated double points, but it cannot have isolated nonnormal double points by Serre's criterion of normality ("normal" is equivalent to "S_2 and R_1") because S has only hypersurface singularities and hence it has only S_2 singularities. Note that $\pi : X \to S$ is the normalization map because it is finite and birational (Zariski Main Theorem ([8, Th. V.4, p. 410])). Hence $\mathrm{Sing}(S)$ is either empty or a curve, i.e. $D = \mathrm{Sing}(S)$. Consider the formulas (i) and (ii) of [14, Prop. 1 at page 211], in which the number, t, of triple points of S is 0 by assumption. For the discussion of the truth of these formulas in our situation, see 3.2 below.

(1) $$c_1(X)^2 = m(m-4)^2 - (3m-16)z - n$$

(2) $$c_2(X) = m(m^2 - 4m + 6) - (3m-8)z - 2n$$

Let C be a general hyperplane section of $X \subset U(1, 3) \subset \mathbf{P}^7$. Hence C is a smooth non degenerate curve of $\mathbf{P}^6$. Set $d := \deg(C)$ and $g := p_a(C)$. We have $d = m+2a = m+8$ and $g = p_g(H)+4 = (m-1)(m-2)/2-z+4$ because the arithmetic genus is the same for all hyperplane sections ([8, Cor. III.9.10]) and for the reducible hyperplane section $H \cup \pi'^{-1}(P)$, $P \in \mathbf{P}^1$, the arithmetic genus is $p_g(H)+4$. Set $m_1 := [(d-1)/6]$, $\varepsilon_1 := := d-6m_1-1$ and $\mu_1 := 1$ if $\varepsilon_1 = 5$ and $\mu_1 := 0$ otherwise. Set $\pi_1(d,6) := := 6m_1(m_1-1)/2 + m_1(\varepsilon_1+1) + \mu_1$. First assume that $g > \pi_1(d,6)$. If $d \geq 20$ by [2, p. 123] (in positive characteristic see for instance the proof of [3, Prop. 2.8] or use [16] and one of the characteristic 0 proofs) C is contained in a minimal degree surface T of $\mathbf{P}^6$; this is the meaning of the integer $\pi_1(d,6)$; curves of degree and genus $\pi_1(d,6)$ on a minimal degree surface T of $\mathbf{P}^6$. By the characteristic free classification of such surfaces T is either a smooth rational scroll or the cone over a rational normal curve of $\mathbf{P}^5$. If T is a cone over a rational normal curve of $\mathbf{P}^5$, every smooth curve of degree ≥ 11 contained in T has every line of the cone as trisecant line. Hence we may assume T smooth, i.e. $T \simeq F_e$ (a Segre-Hirzebruch surface) for some $e \geq 0$. We have $\mathrm{Pic}(F_e) \simeq \mathbf{Z}^2$ with basis h, f such that $h^2 = -e$, $h \cdot f = 1$ and $f^2 = 0$. By the adjunction formula $\omega_T \simeq O_T(-2h + (-e-2)f)$. Since T is a minimal degree surface of $\mathbf{P}^6$, we have $e = 1$ or $e = 3$. Assume $C \in |yh + xf|$. By the structure of curves on F_e we have $x \geq ey > 0$. If $y \geq 3$ all the lines of the ruling of T are trisecant lines of C, contradiction. If $y = 1$, then C is rational, i.e. $g = 0$, contradicting the assumption $g > \pi_1(d,6)$. Hence we may assume $y = 2$. We have $O_T(1) \simeq O_T(h + ((e+5)/2)f)$. Hence $d := m+8 = x+5-e$, i.e. $x = m+3+e$. Using the adjunction formula we obtain $2g-2 = ((x-2-e)f) \cdot (2h+xf)$, i.e. $g = m+1 \leq \pi_1(d,6)$, contradiction.

The inequality $g > \pi_1(d,6)$ is implied by the inequality

(3) $(m+7)^2/12 < (m-2)(m-1)/2 - m(m^2-4m+6)/(3m-8)+4$

which is satisfied if $m \geq 30$, i.e. if $\deg(X) \geq 38$. We claim that $c_2(X) \geq \geq 0$. Let $\kappa(X)$ be the Kodaira dimension of X. Since $\mathrm{char}(\mathbf{K}) \neq 2, 3$, X is not a quasi-elliptic surface in the sense of [5]. Since X has a base point free pencil of smooth elliptic curves or smooth rational curves we have $\kappa(X) \leq 1$ and if $\kappa(X) < 0$ (i.e. X is ruled) then $h^1(X, O_X) \leq 1$, i.e. $\chi(O_X) \geq 0$. Hence the inequality $c_2(X) \geq 0$ follows from Noether's formula $c_1^2(X)+c_2(X) = 12\chi(O_X)$ and the inequality $\chi(O_X) \geq 0$, which in positive characteristic is called Igusa inequality (see [11, last line of p. 294]). Since $c_2(X) \geq 0$ and $n \geq 0$, we have $z(3m-8) \leq m(m^2-4m+6)$

by eq. (2).

Hence we have proved the inequality (3) needed to obtain a contradiction.

(3.2) Here we discuss why we may use formulas (1) and (2) to prove Th. 0.1. The triple point formulas in arbitrary characteristic were proved in [12], [6, Th. 4.4 and §5] and [15]. For a discussion, see also the introduction of [13]. For a discussion of the reason why if there is no triple point then the value of the expected number of triple points given by the enumerative formula is 0, see the discussion in [15, p. 83, lines 6–9] in the framework of curvilinear fibers. However, here the situation is simpler and [12] would be sufficient, because by assumption all the scheme-theoretic fibers of π have length ≤ 2.

(3.3) Here we assume $\deg(\pi) = 2$. Set $S := \pi(X)$. Let $v : S' \to S$ be the normalization of S and $u : X \to S'$ the degree 2 map induced by π, i.e. such that $\pi = u \cdot v$. Since every fiber of v over a smooth point of S' has length 2 (see e.g. [7, Prop. 10.2]), S is locally Cohen-Macaulay and every fiber of π has length ≤ 2, we see that u is bijective. Set $H := u^*(O_S(1))$ and let $\subset S'$ be the image of a fiber of the projection $\pi'|X : X \to \mathbf{P}^1$. Hence, calculating the intersection numbers on the normal surface S', we have $H^2 = \deg(S)$ and (with the notations of 1.2) $a := T \cdot H$. By 2.3 we have $a = 2, 3$ or 4. By Remarks 2.4 and 2.5 we have $a = 4$, but to prove 0.1 this will not be essential. Furthermore, S' is the union of the images of the fibers of π'. Since v has degree 2, for a general $P \in S'$ there are 2 such images. In particular two general such images are not disjoint and meet at least over a smooth point of S', except if π factors through a 2 to 1 map $j' : \mathbf{P}^1 \to \mathbf{P}^1$, i.e. there is an embedding of S' in $U(1,3)$ and a 2 to 1 map $j : U(1,3) \to U(1,3)$ induced by j' and inducing π. This case will be discussed in 3.3.1 below. Here we assume that we are not in this exceptional case. Hence we have $T^2 \geq 1$. By Hodge Index Theorem we have $T^2H^2 \leq (H \cdot T)^2 = a^2$. Hence $\deg(S) = H^2 \leq a^2$. Since $\deg(X) = 2\deg(S) + 2$ by Remark 1.2, we obtain $\deg(X) \leq 34$.

(3.3.1) Here we discuss the exceptional case left open in 3.3, i.e. we assume that for every $P \in \mathbf{P}^1$, there is $\sigma(P) \in \mathbf{P}^1$, $\sigma(P) \neq P$ for general P, and with $\pi"(C'(P)) = \pi"(C'(s(P)))$. This implies that $S := \pi"(X)$ has singular locus finite, because all fibers of $\pi"|X$ have length ≤ 2. Hence S is normal. No such surface has a family of degree 4 smooth elliptic curves with self-intersection 0 and with general member contained in the smooth locus. Hence even this case cannot occur if $\deg(X) \geq 35$.

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