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PSEUDO-ELLIPTIC INTEGRALS AND THE VALUES OF THE WEIERSTRASS ζ-FUNCTION AT TORSION POINTS

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Abstract: A connection is established between early results of Abel and Tchebycheff on pseudo-elliptic integrals and a result of Baker concerning the values of the Weierstrass ζ-function at torsion points.

1. Introduction

Let E be an elliptic curve defined over a number field $K \subset \mathbb{C}$ and $\wp(z)$, $\wp'(z)$ the corresponding Weierstrass functions. We assume that E is given through the usual parameterization

$$\begin{array}{rccc} \text{ParaE}: & \mathbb{C} & \longrightarrow & E \\ & z & \longmapsto & (\wp(z), \wp'(z)) \end{array}$$

such that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{1}$$

where $g_2, g_3 \in K$. We denote the periods of E with ω_1, ω_2 and the lattice generated by the periods with Λ.

Assume that there is a point $z_0 \in \mathbb{C}$ such that $\wp(z_0) \in E(K)$. Then it follows easily from the relation (1) that the values of all derivatives of $\wp(z)$ at z_0 lie in the field K. This is not correct for $\zeta(z_0)$, the value of the Weierstrass ζ-function, the primitive of $-\wp(z)$, at z_0. The value $\zeta(z_0)$ is transcendental by the theorem of Schneider (cf. [9]). However A. Baker proved the following fact (cf. [2], lemma 5): let E be an elliptic curve defined over the number field $K \subset \mathbb{C}$ and let $z_0 = r_1\omega_1 + r_2\omega_2$, $r_1, r_2 \in$ bbq be a torsion point of this curve, then the value $\zeta(z_0) - (r_1\eta_1 + r_2\eta_2)$ where $\eta_i := 2\zeta(\omega_i/2)$, $i = 1, 2$, denote the quasi-periods of E, is an element of K. In particular, there is a canonical splitting $\zeta(z_0) = T(z_0) + A(z_0)$ into a transcendental and an algebraic part.

The purpose of this note is to show how this splitting can be expressed in terms of a pseudo-elliptic integral and to derive an algorithm which allows t he computation of $A(z_0)$ by the means of a continued fraction expansion of an appropriate algebraic function. Our exposition is structured as follows. Section 2 assembles some known results and describes the pseudo-elliptic integrals involved. Section 3 states the precise connection to Weierstrass' ζ-function. In Section 4 the connection proved by Baker is re-established in the context of pseudo-elliptic integrals using an early result by Abel. In Section 5 we present a computed example.

2. The integrals

Let $\wp(z)$, $\zeta(z)$, $\sigma(z)$ be the Weierstrass elliptic functions belonging to the lattice Λ generated by $\omega_1, \omega_2 \in \mathbb{C}$. Let us recall the following relations (cf. [6], p. 239):

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}, \tag{2}$$

$$(\log \sigma(u))' = \frac{\sigma'(u)}{\sigma(u)} = \zeta(u). \tag{3}$$

Let an elliptic curve be given by the equation

$$y^2 = 4x^3 - g_2 x - g_3, \tag{4}$$

together with a point $z_0 \in \mathbb{C}$ such that $(x_0, y_0) := (\wp(z_0), \wp'(z_0)) \in E(K)$, i.e., is a K-rational point.

By means of the rational transformation

$$\xi = \frac{1}{2}\frac{y-y_0}{x-x_0}, \qquad \eta = 2x + x_0 - \frac{1}{4}\left(\frac{y-y_0}{x-x_0}\right)^2$$

one obtains a singular model S_K defined over the field K. We say this model is generated by the point z_0 and write sometimes $S_K(z_0)$. This model is parameterized by the function

$$\begin{array}{rccc} \mathrm{ParaS}: & \mathbb{C} & \longrightarrow & S_K(z_0) \\ & z & \longmapsto & (P(z), P'(z)) \end{array}$$

where

$$P(z) = \frac{1}{2}\frac{\wp'(z) - \wp'(z_0)}{\wp(z) - \wp(z_0)}.$$

$S_K(z_0)$ is the zero set of the equation

$$\eta^2 = R_{z_0}(\xi) := \xi^4 + c_2\xi^2 + c_3\xi + c_4 \tag{5}$$

where

$$c_2 = -6x_0, \qquad c_3 = 4y_0, \qquad c_4 = g_2 - 3x_0^2.$$

The singular locus of S_K is the image of the points $0, z_0 \in \mathbb{C}$ by ParaS :, the points at infinity.

Elliptic integrals of the form

$$\mathrm{Int} := \int \frac{\xi + A}{\sqrt{R_{z_0}(\xi)}} d\xi \tag{6}$$

which are defined on S_K were extensively studied in the last century (cf. [10], [11]]): Tchebycheff was able to reduce the problem of integration in finite terms (cf. [4], [7]) for elliptic integrals to the question of pseudo-ellipticity of integrals of the form (6): Int is pseudo-elliptic if there exist $p(\xi), q(\xi) \in \mathbb{C}[\xi]$ such that

$$\mathrm{Int} = \frac{1}{\lambda}\log\frac{p(\xi) - q(\xi)\sqrt{R_{z_0}(\xi)}}{p(\xi) + q(\xi)\sqrt{R_{z_0}(\xi)}} \tag{7}$$

for some $\lambda \in \mathbb{Z}$ (cf. [3]).

It is easy to see that for a given $R_{z_0}(\xi)$ at most one value A exists such that (7) holds (cf. [10], p. 2).

We recall some facts about continued fraction expansion (cf. [5], p.84). Let $\alpha_0 = \sum_{m \geq m_0} \gamma_m t^{-m}$ be the Laurent expansion of $\sqrt{R_{z_0}}$ at a point p at infinity. One puts $a_1 := [\alpha_0] := \sum_{0 \geq m \geq m_0} \gamma_m t^{-m}$ and $a_i := = [\alpha_{i-1}]$ with α_i the Laurent expansion of $\frac{1}{\alpha_{i-1} - [\alpha_{i-1}]}$ for $i \geq 1$. The sequence $\{a_i\}_{i=1}^{\infty}$ is called the continued fraction expansion of α_0 at the point p. One puts as usual

$$P_0 := 1, \qquad P_1 := a_1, \qquad P_i := a_i P_{i-1} + P_{i-2}$$
$$Q_0 := 0, \qquad Q_1 := 1, \qquad Q_i := a_i Q_{i-1} + Q_{i-2}.$$

The continued fraction expansion $\{a_i\}_{i=1}^{\infty}$ is called pseudo-periodic if there exists a $k \in \mathbb{N}^*$ such that

$$P_k^2 - Q_k^2 R_{z_0} = c \in K.$$

The smallest $k \in \mathbb{N}^*$ with this property is called the pseudo-period.

The following proposition summarizes various known results (cf. [8], p.296; [11], p.105; [5], p.90).

Proposition 2.1. *The following conditions for* Int $= \int \frac{\xi + A}{\sqrt{R_{z_0}(\xi)}} d\xi$ *are equivalent*

a) *the continued fraction expansion of $\sqrt{R_{z_0}}$ at one of the points at infinity (hence on both) is pseudo-periodic with pseudo-period $l-1$;*

b) *there exists a value A such that (7) holds for appropriate $p(\xi)$, $q(\xi) \in K[\xi]$ of* $\deg p = l$ *and* $\deg q = l-2$*, with $\lambda = 2l$;*

c) *z_0 is a torsion point of order l.*

Remark 2.2. If Int is pseudo-elliptic and z_0 is a torsion point of order l then $p(\xi) = P_{l-1}(\xi)$ and $q(\xi) = Q_{l-1}(\xi)$.

3. The splitting

Proposition 3.1. *Let S_K be generated by the torsion point z_0 of order l. If $lz_0 = n_1\omega_1 + n_2\omega_2 \in \Lambda$, then $\zeta(z_0) = T(z_0) + A(z_0)$ where $lT(z_0) = n_1\eta_1 + n_2\eta_2$, and $A(z_0) = A$ where A is the unique value such that $\int \frac{\xi + A}{\sqrt{R_{z_0}(\xi)}} d\xi$ is pseudo-elliptic.*

Proof. Applying (2) we obtain by elementary calculations:

$$\begin{aligned}
\text{Int} &= \int \left(\frac{1}{2} \frac{\wp'(z) - \wp'(z_0)}{\wp(z) - \wp(z_0)} + A \right) dz = \\
&= \int \big(\zeta(z + z_0) - \zeta(z) - \zeta(z_0) + A \big) dz = \\
&= \int \big(\zeta(z + z_0) - \zeta(z) \big) dz + \big(A - \zeta(z_0) \big) z.
\end{aligned}$$

Hence putting

$$\Sigma(z) =: \exp(2l \cdot \mathrm{Int}),$$

(3) implies that

$$\Sigma(z) = \exp(2l \cdot \int ((\zeta(z+z_0) - \zeta(z))dz) \cdot \exp(2l \cdot (A - \zeta(z_0))z) =$$

$$= \frac{\sigma(z+z_0)^{2l}}{\sigma(z)^{2l}} \cdot \exp(2l \cdot (A - \zeta(z_0))z).$$

Since z_0 is a torsion point, $\Sigma(z)$ must be a doubly periodic function of z by Prop. 2.1: $\Sigma(z+\omega) = \Sigma(z)$. This imposes a condition on the expression $A - \zeta(z_0)$:

We have $\sigma(z+\omega) = \sigma(z)e^{\eta z + c}$, where $c \in \mathbb{C}$ (cf. [6]). Now

$$\Sigma(z+\omega) = \frac{\sigma(z+\omega+z_0)^{2l}}{\sigma(z+\omega)^{2l}} \cdot \exp(2l \cdot (A - \zeta(z_0))(z+\omega)) =$$

$$= \frac{\sigma(z+z_0)^{2l} \exp(2l\eta(z+z_0)+2lc)}{\sigma(z)^{2l} \exp(2l\eta z+2lc)} \cdot \exp(2l \cdot (A - \zeta(z_0))(z+\omega)) =$$

$$= \Sigma(z) \cdot N$$

where

$$N := \exp(2l \cdot \eta \cdot z_0 + 2l \cdot (A - \zeta(z_0)) \cdot \omega).$$

This implies that $N = 1$. Hence there is a value $m(\omega) \in \mathbb{Z}$ such that

$$\zeta(z_0) - A = \frac{l \cdot z_0 \cdot \eta - \pi i \cdot m(\omega)}{l\omega}. \tag{8}$$

We determine the number $m(\omega)$: the expression $\zeta(z_0) - A$ is independent of ω and η. We substitute ω_1, η_1 and then ω_2, η_2 in equation (8):

$$\frac{lz_0\eta_1 - \pi i m(\omega_1)}{l\omega_1} = \frac{lz_0\eta_2 - \pi i m(\omega_2)}{l\omega_2}.$$

Applying the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ (cf. [6], p.241) we obtain

$$2lz_0 = m(\omega_1)\omega_2 - m(\omega_2)\omega_1.$$

This implies that

$$n_1 = -m(\omega_2)/2, n_2 = m(\omega_1)/2$$

and that

$$\zeta(z_0) - A = \frac{(n_1\omega_1 + n_2\omega_2)\eta_1 - 2\pi i n_2}{l\omega_1} = \frac{n_1\eta_1 + n_2\eta_2}{l} = T(z_0). \qquad \diamond$$

Example. Let $z_0 = \omega_1/2$. Hence $l = 2$, and $n_1 = 1, n_2 = 0$. This implies that $\zeta(\omega_1/2) - A = \eta_1/2$. By definition $\eta_1 = 2\zeta(\omega_1/2)$ and hence $A = 0$. This implies that the integral $\int \frac{\xi}{\sqrt{R_{\omega_1/2}(\xi)}} d\xi$ on $S_K(\omega_1/2)$ is pseudo-elliptic.

4. The algebraic part

The following proposition reveals the algebraic nature of $A(z_0)$ and re-establishes Baker's result.

Proposition 4.1. *Let E/K be an elliptic curve defined over $K \subset \subset \mathbb{C}$ and $p_0 = (\wp(z_0), \wp'(z_0))$ a K-rational torsion point of E. Then $A(z_0) \in K$.*

Proof. We expand $\sqrt{R_{z_0}(\xi)}$ at one of the places at infinity in a Laurent series. This series is an element of $K((t))$, where t is a uniformizing parameter, since the coefficient of ξ^4 equals 1. We calculate the continued fraction expansion $\{a_i\}_{i=1}^{\infty}$ of the Laurent series. Assume that z_0 is a torsion point of order l. By Prop. 2.1 the continued fraction expansion is pseudo-periodic with pseudo-period $l-1$. Hence

$$P_{l-1}^2 - Q_{l-1}^2 R_{z_0}$$

is a constant. Applying the following result of Abel (cf. [1], p. 106) about the connection between P_{l-1}, Q_{l-1} and the nominator of the integrand we obtain finally

$$x + A = 2(P_{l-1}Q'_{l-1} - Q_{l-1}P'_{l-1})R_{z_0} + P_{l-1}Q_{l-1}R'_{z_0} \in K[x],$$

and hence $A \in K$. $\Diamond$

Remark 4.2. By the theorem of Schneider mentioned in the introduction it follows now that the expression $T(z_0)$ is transcendental.

5. Examples

Baker's formula (cf. [2] p. 148) expresses A in terms of $\wp(mz_0)$, $\wp'(mz_0)$, $m = 2, \ldots l-1$, which can be computed from $g_2, g_3, \wp(z_0)$. The proof of Prop. 4.1 yields a more local algorithm to compute the value $A(z_0)$ from the data $g_2, g_3, \wp(z_0)$. We used this algorithm to calculate in the following ex ample for some torsion points the corresponding pseudo-elliptic integrals, i.e., c_2, c_3, c_4 and $A(z_0)$.

Let the following elliptic curve over $\mathbb{Q}$ be given $E\colon y^2 = 4x^3 - -172x + 664$. Its rational Mordell-Weil group has a subgroup of order 7. Hence we find 6 torsion points $z_0 = (x, y) \neq \infty$ defined over $\mathbb{Q}$. The following table gives the coordinates of the torsion points, the coefficient of the corresponding equation of degree 4 for the singular model $S_K(z_0)$ and the algebraic part $A(z_0)$.

(x,y)	$(3,16)$	$(-5,-32)$	$(11,-64)$	$(11,64)$	$(-5,32)$	$(3,-16)$
c_2	-18	30	-66	-66	30	-18
c_3	64	-128	-256	256	128	-64
c_4	145	97	-191	-191	97	145
$A(z_0)$	$1/7$	$-5/7$	$17/7$	$-17/7$	$5/7$	$-1/7$

Computations were performed using the symbolic computer algebra systems Mathematica and Maple.

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