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ALIASING IN THE SAMPLING RESTORATION OF NON-BAND-LIMITED HOMOGENEOUS RANDOM FIELDS

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Abstract: Spectral representations of the multiple sampling cardinal series expansion for a non-band-limited homogeneous random field are established. With the help of obtained representations mean-square aliasing error upper bounds are derived and the structure of the aliasing error is discussed.

1. Introduction

The spectral representation of a random signal plays a very useful role in the sampling restoration error analysis since the Hilbert-space background and the related correlation theory of homogeneous random fields. The well-known isometry between the Hilbert-space of the considered field and the L_2-space of the functions integrable with respect to the corresponding spectral measure ensures the easy derivation of the upper bound for the mean-square aliasing error in the sampling restoration procedure with the usual multiple Kotel'nikov–Shannon formula.

The mean-square restoration problem connected to the aliasing error of non-band-limited (NBL) stochastic signals by its sampling cardinal series (Kotel'nikov-formula) has been considered by Brown (1978), Pogány (1995), Pogány and Peruničić (1992) (weakly stationary stochastic processes); Kambo and Mehta (1980), Habib and Cambanis (1981) (harmonizable random processes); Pogány (1991; 1993), (homogeneous random fields). Spectral (or Fourier integral-form) representations of the sampling cardinal series expansion (SCSE) for random signals are obtained in Habib and Cambanis (1981) with sufficient conditions and in Pogány (1995) by sufficient and necessary conditions upon the spectral measure of the considered stochastic process. At this point we have to remark that all this restoration and spectral representation formulas deal with the uniform sampling case.

The main goal of this paper is to establish such representation for multiple SCSE for the NBL homogeneous random fields (HRF). Also we discuss the convergence problems of the multiple SCSE when the bandwidth vector W increases to infinity and we obtain mean-square aliasing error upper upper bounds, generalizing in the same way the well-known Brown bound to the multidimensional case.

2. Preliminary definitions and results

We define on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ the complex random field as a random function $\xi(\omega, \mathbf{x}) : \ \Omega \times \mathcal{X} \longrightarrow \mathbf{C}, \ \mathcal{X} \subseteq \mathbf{R}^r$. If $r = 1$ then $\xi(\mathbf{x}) := \xi(\omega, \mathbf{x})$ is a stochastic process. If $r > 1$ then $\xi(\mathbf{x})$ is an r-dimensional random field (RF). Troughout this paper we will consider random fields with the finite second moment, i.e. we assume $\mathbf{E}|\xi(\mathbf{x}|^2 < \infty$.

Let us consider an r-dimensional RF $\{\xi(\mathbf{x}) | \ \mathbf{x} \in \mathbf{R}^r\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The RF $\xi(\mathbf{x})$ is said to be *homogeneous* if its mathematical expectation $\mathbf{E}\xi(\mathbf{x})$ is constant (zero, for simplicity[1]) and when its correlation function $\mathcal{K}(\mathbf{x}, \mathbf{y}) := \mathbf{E}\xi(\mathbf{x})\xi^*(\mathbf{y})$ depends only on the difference $\mathbf{x} - \mathbf{y} = (x_1 - y_1, \cdots, x_r - y_r)$, i.e. $\mathcal{K}(\mathbf{x}, \mathbf{y}) = \mathcal{K}(\mathbf{x} - \mathbf{y}, 0) = \mathcal{K}(\mathbf{x} - \mathbf{y})$.[2]

The value $\mathcal{K}(0)$ is the variance of the HRF $\xi(\mathbf{x})$, i.e. $\sigma^2\xi(\mathbf{x}) = \mathbf{E}|\xi(\mathbf{x}|^2 = \mathcal{K}(0)$. Since we concentrate our attention to the non-band-

[1]If $\mathbf{E}\xi(\mathbf{x}) \neq 0$ then the HRF $\xi_0(\mathbf{x}) := \xi(\mathbf{x}) - \mathbf{E}\xi(\mathbf{x})$ has the mathematical expectation equal to zero; the analysis entails no restriction and ξ_0 contains all informations about the field ξ.

[2]The asterisk * denotes complex conjugation.

limited HR case, the field $\xi(\mathbf{x})$ and its correlation function possess the following spectral representations:

$$(1) \qquad \xi(\mathbf{x}) = \int\limits_{\mathbf{R}^r} e^{i\langle \mathbf{x},\lambda\rangle} d\mathbf{Z}(\lambda); \quad \mathcal{K}(\mathbf{x}) = \int\limits_{\mathbf{R}^r} e^{i\langle \mathbf{x},\lambda\rangle} d\Phi(\lambda),$$

where $\langle \mathbf{x}, \lambda\rangle := \sum_{k=1}^{k=r} x_k\lambda_k$ is the inner product of the r-dimensional vectors $\mathbf{x}$ and λ; $\mathbf{Z}(\lambda)$ is the random spectral field and $\Phi(\lambda)$ denotes the spectral distribution function of $\xi(\mathbf{x})$. Moreover $d\Phi(\lambda) = \mathbf{E}|d\mathbf{Z}(\lambda)|^2$ and $\mathbf{Z}(\lambda)$ possesses orthogonal increments, see Yaglom (1987a; 327–329).

The sampling cardinal series expansion (SCSE in the sequel) $\xi_W(\mathbf{x})$ with respect to the given choice of bandwidth-vector $W = (w_1, \cdots, w_r)$; $w_j > 0, j = \overline{1,r}$ for the considered NBL HRF $\xi(\mathbf{x})$ is defined as

$$(2) \qquad \xi_W(\mathbf{x}) := \sum_{j=1}^{j=r} \sum_{n_j=-\infty}^{n_j=\infty} \xi(x^n) \prod_{k=1}^{k=r} \operatorname{sinc}(w_k x_k - n_k\pi),$$

where $\operatorname{sinc}(t) := \frac{\sin t}{t}$ for $t \neq 0$ and $\operatorname{sinc} 0) := 1$; the quantity x^n runs over the lattice

$$\mathcal{L}^{(1)}(W) := \left\{ \left(\frac{n_1\pi}{w_1}, \cdots, \frac{n_r\pi}{w_r} \right) \mid n_j \in \mathbf{Z} \right\}.$$

The truncated sampling expansion $\xi_{W,N}(\mathbf{x})$ with respect to the given W and to the given coordinatewise sampling sizes $N_j,\ j = \overline{1,r}$ is defined by the expression

$$(3) \qquad \xi_{W,N}(\mathbf{x}) = \sum_{j=1}^{j=r} \sum_{|n_j|\le N_j} \xi(x^n) \prod_{k=1}^{k=r} \operatorname{sinc}(w_k x_k - n_k\pi).$$

Here $x^n \in \mathcal{L}_N^{(1)}(W) := \left\{ \left(\frac{n_1\pi}{w_1}, \cdots, \frac{n_r\pi}{w_r} \right) \mid \ |n_j| \le N_j, j = \overline{1,r} \right\}$ and N denotes the multiindex $(N_1, \cdots, N_r)$.

The $2w$-periodic extension of the Fourier-kernel $e^{it\lambda}$ from $(-w, w)$ to the whole real axis possesses the Fourier-series

$$\sum_{n=-\infty}^{n=\infty} e^{in\frac{\pi}{w}\lambda} \operatorname{sinc}(wt - n\pi).$$

Now, it follows by standard Fourier-theory results that for any fixed t and w

$$(4) \qquad \sum_{n=-\infty}^{n=\infty} e^{in\frac{\pi}{w}\lambda} \operatorname{sinc}(wt - n\pi) = \begin{cases} e^{it(\lambda - 2kw)} & (2k-1)w < \lambda < (2k+1)w \\ \cos(wt) & \lambda = (2k+1)w, \end{cases}$$

for all λ boundedly, see e.g. Habib and Cambanis (1981; pp. 145–146). Let us denote the function (4) by $A_w(t, \lambda)$.

Similarly, we get the multiple Fourier-series of the r-variate Fourier-kernel $e^{i\langle \mathbf{x},\lambda\rangle}$ for fixed bandwidth $W > 0$ and for fixed time-vector $\mathbf{x}$ as follows

$$\sum_{j=1}^{j=r}\sum_{n_j=-\infty}^{n_j=\infty} e^{in_j\frac{\pi}{w_j}\lambda_j}\prod_{j=1}^{j=r}\operatorname{sinc}(w_jx_j-n_j\pi).$$

This Fourier-series converges coordinatewise boundedly with respect to λ to the depending functions $A_{w_j}(x_j,\lambda_j)$; $j=\overline{1,r}$. Thus, according to the already used notations we get

$$\sum_{j=1}^{j=r}\sum_{n_j=-\infty}^{n_j=\infty} e^{in_j\frac{\pi}{w_j}\lambda_j}\prod_{k=1}^{k=r}\operatorname{sinc}(w_kx_k-n_k\pi)=\prod_{j=1}^{j=r}A_{w_j}(x_j,\lambda_j):=\mathcal{A}_W(\mathbf{x},\lambda). \tag{5}$$

The so-called *mean-square aliasing error* $\mathcal{E}_W(\mathbf{x})$ we define as the r-variate time function $\mathbf{E}|\xi(\mathbf{x})-\xi_W(\mathbf{x})|^2$.

Finally let $\mathcal{H}(\xi)$ be the Hilbert-space of the NBL HRF $\xi(\mathbf{x})$, i.e. $\mathcal{H}(\xi)$ is the linear mean-square span of the set $\{\xi(\mathbf{x})|\ \mathbf{x}\in\mathbf{R}^r\}$. Put $L_2(d\Phi;\mathbf{R}^r)$ for the space of all functions, square-integrable on $\mathbf{R}^r$ with respect to the measure $d\Phi$, i.e. $L_2(d\Phi;\mathbf{R}^r):=\{\varphi|\ \int_{\mathbf{R}^r}|\varphi(\lambda)|^2d\Phi(\lambda)\}$. Then there exists the well-known isometry $\xi(\mathbf{x})\leftrightarrow e^{i\langle\mathbf{x},\lambda\rangle}$ between $\mathcal{H}(\xi)$ and $L_2(d\Phi;\mathbf{R}^r)$ realized by the mathematical expectation, or in other words

$$\mathbf{E}|\xi(\mathbf{x})-\xi(\mathbf{y})|^2=\int_{\mathbf{R}^r}\left|e^{i\langle\mathbf{x},\lambda\rangle}-e^{i\langle\mathbf{y},\lambda\rangle}\right|^2d\Phi(\lambda). \tag{6}$$

By this result we will derive our principal efforts.

3. General results

At first we will establish a spectral (Fourier-integral form) representation for the SCSE of the NBL HRF $\xi(\mathbf{x})$, sampled equidistantly with respect to the given bandwidth vector $W=(w_1,\cdots,w_r)>0$ coordinatewise.

Theorem 1. *Let $\{\xi(\mathbf{x})|\ \mathbf{x}\in\mathbf{R}^r\}$ be an r-dimensional NBL HRF with random spectral field $\mathbf{Z}$, spectral distribution function Φ and with the multidimensional SCSE $\xi_W(\mathbf{x})$. Then for all fixed $\mathbf{x}\in\mathbf{R}^r$ and $W>0$, we have*

$$\xi_W(\mathbf{x}):=\sum_{j=1}^{j=r}\sum_{n_j=-\infty}^{n_j=\infty}\xi(x^n)\prod_{k=1}^{k=r}\operatorname{sinc}(w_kx_k-n_k\pi)=\int_{\mathbf{R}^r}\mathcal{A}_W(\mathbf{x},\lambda)d\mathbf{Z}(\lambda), \tag{7}$$

where the equality is with the probability 1 and the series $\xi_W(\mathbf{x})$ converges in the mean-square. Also

(8) $$\mathcal{K}_W(\mathbf{x}) := \mathbf{E}\xi_W(\mathbf{x})\xi_W^*(0) = \int_{\mathbf{R}^r} \mathcal{A}_W(\mathbf{x},\lambda)d\Phi(\lambda),$$

where $\mathcal{K}_W(\mathbf{x})$ is the correlation function[3] *of the SCSE $\xi_W(\mathbf{x})$.*
Proof. According to the already noted isometry between $\mathcal{H}(\xi)$ and $L_2(d\Phi;\mathbf{R}^r)$, the quoted Fourier-series results ensure, by (6) and the dominated convergence theorem, that

$$\mathbf{E}|\xi_{W,N}(\mathbf{x}) - \int_{\mathbf{R}^r} \mathcal{A}_W(\mathbf{x},\lambda)d\mathbf{Z}(\lambda)|^2 =$$

$$= \int_{\mathbf{R}^r} \left|\sum_{j=1}^{j=r} \sum_{|n_j|\le N_j} e^{in_j\frac{\pi}{w_j}\lambda_j} \prod_{k=1}^{k=r} \mathrm{sinc}(w_k x_k - n_k\pi) - \mathcal{A}_W(\mathbf{x},\lambda)\right|^2 d\Phi(\lambda) \longrightarrow 0,$$

as $N^* = \min_{1\le j\le r} N_j \to \infty$, since the series (5) converges boundedly, and Φ has finite total variation $|\Phi|$ ($\xi(\mathbf{x})$ is with the finite second moment). This shows (7). Using the derived spectral representation (7) we have

$$\mathcal{K}_W(\mathbf{x}) = \mathbf{E}\xi_W(\mathbf{x})\xi_W^*(0) =$$
$$= \int_{\mathbf{R}^r}\int_{\mathbf{R}^r} \mathcal{A}_W(\mathbf{x},\lambda)\mathcal{A}_W^*(0,\lambda)\mathbf{E}|d\mathbf{Z}(\lambda)|^2 = \int_{\mathbf{R}^r} \mathcal{A}_W(\mathbf{x},\lambda)d\Phi(\lambda),$$

since $\mathcal{A}_W(0,\lambda) \equiv 1$, compare (4). ◊

Now, we are interested in the mean-square restoration error upper bound in approximating the NBL HRF $\xi(\mathbf{x})$ by its SCSE $\xi_W(\mathbf{x})$ for the given choice of the bandwidth-vector $W > 0$.

Corollary 1. *Let $\xi(\mathbf{x})$ be the same as in the previous theorem. Then we have uniformly in $\mathbf{x}$ that*

(9) $$\xi(\mathbf{x}) = \lim_{w^*\to\infty} \xi_W(\mathbf{x}).$$

Here the equality holds in the mean-square sense; $w^* := \min_{1\le j\le r} w_j$.
Proof. At first consider the mean-square aliasing error

(10) $$\mathcal{E}_W(\mathbf{x}) = \mathbf{E}|\xi(\mathbf{x}) - \xi_W(\mathbf{x})|^2 = \int_{\mathbf{R}^r} \left|e^{i\langle\mathbf{x},\lambda\rangle} - \mathcal{A}_W(\mathbf{x},\lambda)\right|^2 d\Phi(\lambda).$$

Denotes $\mathcal{W} := \times_{j=1}^{j=r}(-w_j,w_j)$. Because of $\mathcal{A}_W(\mathbf{x},\lambda)$ coincides on $\mathcal{W}$ with $e^{i\langle\mathbf{x},\lambda\rangle}$ the relation (10) becomes

[3]The correlation function coincides with the covariance function for the mean zero random functions.

$$\mathcal{E}_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r \setminus \mathcal{W}} \left| e^{i\langle \mathbf{x}, \lambda \rangle} - \mathcal{A}_W(\mathbf{x}, \lambda) \right|^2 d\Phi(\lambda) \le 4|\Phi|(\mathbf{R}^r \setminus \mathcal{W}). \tag{11}$$

Denote by $\mathcal{B}(\mathbf{R}^r)$ the Borel σ-field of $\mathbf{R}^r$. As $\Phi(\cdot)$ is a finite measure of $\mathbf{R}^r$ and $\Phi(B) = \int\limits_B d\Phi(\lambda),\ B \in \mathcal{B}(\mathbf{R}^r)$, by putting $\mathcal{P}(B) = \frac{1}{\mathcal{K}(0)} \int\limits_B d\Phi(\lambda)$ we get a probability measure. Since it is vanishing with respect to the decreasing sequences of events, by the evaluation (11) it immediately follows that $\mathcal{E}_W(\mathbf{x}) \to 0$ as w^* increases to infinity. ◊

Remark 1. The inequality

$$\mathcal{E}_W(\mathbf{x}) = \mathbf{E}|\xi(\mathbf{x}) - \xi_W(\mathbf{x})|^2 \le 4|\Phi|(\mathbf{R}^r \setminus \mathcal{W})$$

is a generalization of the multidimensional variant of the well-known Brown mean-square aliasing error upper bound, compare with Brown (1978). Moreover, we have the following robust improvement of this inequality. The mixed-exponent $|s| = \sum_{j=1}^{j=r} s_j$, say, s_j nonnegative integers, i.e. for some $\alpha = (\alpha_1, \cdots, \alpha_r)$ it is $\alpha^{\gamma|s|} := \prod_{j=1}^{j=r} \alpha_j^{\gamma s_j},\ \gamma \in \mathbf{R}$. Then, if $\xi(\mathbf{x})$ possesses $|s| > 0$ mean-square derivatives, it follows that

$$\mathcal{E}_W(\mathbf{x}) \le \frac{4}{W^{2|s|}} \int\limits_{\mathbf{R}^r \setminus \mathcal{W}} \lambda^{2|s|} d\Phi(\lambda) \le \frac{4}{W^{2|s|}} \left| \frac{\partial^{2|s|}\mathcal{K}(0)}{\partial x_1^{2s_1} \cdots \partial x_r^{2s_r}} \right|, \tag{12}$$

where the constant 4 is sharp, compare with Pogány (1993; Th. (ii)).

In the sequel we will be interested in the structure of the mean-square aliasing error $\mathcal{E}_W(\mathbf{x})$. Notice that $\langle a, b \rangle = \sum_{j=1}^{j=r} a_j b_j$, the inner product on $\mathbf{R}^r$. Without changing the order of coordinates of the vector $a = (a_1, \cdots, a_r)$, we choose q coordinates from a, $0 \le q \le r$, $a^r \equiv \equiv a$; $a^0 = (0, \cdots, 0)_{1 \times r}$. Let a^q denotes the new vector which consists of such q coordinates, and let $\underline{a}^{r-q}$ be its *"complement"* with respect to a. Thus $\langle a^q, b^q \rangle = \sum_{j=1}^{j=q} a_{k_j} b_{k_j}$, where $k_j \in \{1, \cdots, q\}$. Also put $\langle \mathbf{m}a^p, b^p \rangle = \sum_{j=1}^{j=p} m_j a_j b_j$ for the p-tuple of integers $\mathbf{m} = (m_1, \cdots, m_p)$. It will be not hard to recognize the difference between the partial-time vector $\mathbf{x}^p$ and the lattice point $x^n \in \mathcal{L}^{(1)}(W)$.

Theorem 2. *Let a NBL HRF $\{\xi(\mathbf{x})|\ \mathbf{x} \in \mathbf{R}^r\}$ the same as in the* Th. 1 *and let $\mathbf{l} = (l_1, \cdots, l_r)$ be the permutation of $(1, \cdots, r)$ defined by choosing W^q, $\underline{W}^{r-q}$; also put $\mathbf{k} = (k_1, \cdots, k_r)$. Then:*

$$\mathcal{E}_W(\mathbf{x}) \le \tag{13}$$

$$\le \sum_{\mathbf{k}:\ k_1, \cdots, k_r \neq 0} \sum_{q=0}^{q=r} \left(1 + \prod_{j=q+1}^{j=r} \cos w_{l_j} x_{l_j} \right)^2 |\Phi|(C_{r,q}(\mathbf{k}), (2\mathbf{k}+1)\underline{W}^{r-q})$$

where

$$|\Phi|(C_{r,q}(\mathbf{k}),(2\mathbf{k}+1)\underline{W}^{r-q}) = \int\limits_{(2k_1-1)w_{l_1}+}^{(2k_1+1)w_{l_1}-} \cdots \int\limits_{(2k_q-1)w_{l_q}+}^{(2k_q+1)w_{l_q}-} d\Phi(\lambda^q,(2\mathbf{k}+1)\underline{W}^{r-q}),$$

and $C_{r,q}(\mathbf{k}) = \times_{j=1}^{j=q}\left((2k_j-1)w_{l_j},(2k_j+1)w_{l_j}\right)$ *is the q-dimensional open rectangle where the integration is done.*

Proof. According to the definition of the function $\mathcal{A}_W(\mathbf{x},\lambda)$ we need the decomposition

$$\mathbf{R} = \sum_{k_j=-\infty}^{k_j=\infty}\left[\left((2k_j-1)w_{l_j},(2k_j+1)w_{l_j}\right)+\{(2k_j+1)w_{l_j}\}\right].$$

Bearing in mind the above introduced notations, we have

$$\mathcal{E}_W(\mathbf{x}) = \sum_{k_1,\cdots,k_r}\sum_{q=0}^{q=r}\int\limits_{(2k_1-1)w_{l_1}+}^{(2k_1+1)w_{l_1}-}\cdots\int\limits_{(2k_q-1)w_{l_q}+}^{(2k_q+1)w_{l_q}-}\left|e^{i\langle\mathbf{x}^q,\lambda^q\rangle}e^{i\langle\underline{\mathbf{x}}^{r-q},(2\mathbf{k}+1)\underline{W}^{r-q}\rangle} - \mathcal{A}_W(\mathbf{x}^q,\underline{\mathbf{x}}^{r-q},\lambda^q,(2\mathbf{k}+1)\underline{W}^{r-q})\right|^2 d\Phi(\lambda^q,(2\mathbf{k}+1)\underline{W}^{r-q}). \tag{14}$$

Since $e^{i\langle\mathbf{x},\lambda\rangle}$ coincides with $\mathcal{A}_W(\mathbf{x},\lambda)$ on the principal rectangle $\mathcal{W} = \times_{j=1}^{j=r}(-w_j,w_j)$ we have

$$\begin{aligned}\mathcal{E}_W(\mathbf{x}) &= \sum_{\mathbf{k}:\,k_1,\cdots,k_r\neq 0}\sum_{q=0}^{q=r}\left|1-e^{i(\langle\mathbf{x},2\mathbf{k}W\rangle+\langle\underline{\mathbf{x}}^{r-q},\underline{W}^{r-q}\rangle)}\prod_{j=q+1}^{j=r}\cos w_{l_j}x_{l_j}\right|^2\times\\ &\quad\times|\Phi|(C_{r,q}(\mathbf{k}),(2\mathbf{k}+1)\underline{W}^{r-q}) =\\ &= \sum_{\mathbf{k}:\,k_1,\cdots,k_r\neq 0}\sum_{q=0}^{q=r}\left[\left(1-\prod_{j=q+1}^{j=r}\cos w_{l_j}x_{l_j}\right)^2+4\prod_{j=q+1}^{j=r}\cos w_{l_j}x_{l_j}\times\right.\\ &\quad\left.\times\sin^2\frac{\langle\mathbf{x},2\mathbf{k}W\rangle+\langle\underline{\mathbf{x}}^{r-q},\underline{W}^{r-q}\rangle}{2}\right]|\Phi|(C_{r,q}(\mathbf{k}),(2\mathbf{k}+1)\underline{W}^{r-q}).\end{aligned} \tag{15}$$

Now, the estimate (13) is the immediate consequence of (15). ◊

We are also interested in the exact form of the mean-square aliasing error and its upper bound (13) under some restrictions upon the spectral distribution function $\Phi(\lambda)$ of the considered HRF $\xi(\mathbf{x})$.

Theorem 3. *If* $|\Phi|(C_{r,q}(\mathbf{k}),(2\mathbf{k}+1)\underline{W}^{r-q}) = 0$ *for all* $0\le q\le r-1$, *then we have*

$$\xi_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r} \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} d\mathbf{Z}(\lambda), \quad \mathcal{K}_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r} \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} d\Phi(\lambda), \tag{16}$$

where $\left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W}$ *stays for the* $2W$*-periodic extension of* $e^{i\langle \lambda,\mathbf{x}\rangle}$ *from* $-W < \lambda \leq W$ *to* $\mathbf{R}^r$. *Also we have*

$$\mathcal{E}_W(\mathbf{x}) = 4 \sum_{\mathbf{k}:\ k_1,\cdots,k_r\neq 0} \sin^2\langle \mathbf{x}, \mathbf{k}W\rangle |\Phi|(C_{r,r}(\mathbf{k})). \tag{17}$$

Proof. The assumption $|\Phi|(C_{r,q}(\mathbf{k}), (2\mathbf{k}+1)\underline{W}^{r-q}) = 0$, $q \in \{1, \cdots, r-$ $-1\}$ means a kind of "continuity" of Φ on the whole lattice $\{(2\mathbf{k}+1)W :$ $:= ((2k_1+1)w_1, \cdots, (2k_r+1)w_r)|\ k_j \text{ integers}\}$. This gives us, with the help of the isometry (6), the following relation:

$$\mathbf{E}\left|\int\limits_{\mathbf{R}^r} \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} d\mathbf{Z}(\lambda) - \int\limits_{\mathbf{R}^r} \mathcal{A}_W(\mathbf{x},\lambda) d\mathbf{Z}(\lambda)\right)|^2 =$$

$$= \int\limits_{\mathbf{R}^r} \left|\left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} - \mathcal{A}_W(\mathbf{x},\lambda)\right|^2 d\Phi(\lambda) \equiv 0,$$

while $\left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} \equiv \mathcal{A}_W(\mathbf{x},\lambda)$ on $\lambda \in \mathbf{R}^r \setminus \{(2\mathbf{k}+1)W\}$. Therefore it follows that

$$\xi_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r} \mathcal{A}_W(\mathbf{x},\lambda) d\mathbf{Z}(\lambda) = \int\limits_{\mathbf{R}^r} \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} d\mathbf{Z}(\lambda)$$

in the mean-square. Now, the spectral representation $\xi_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r} (e^{i\langle \mathbf{x},\lambda\rangle})_{2W} d\mathbf{Z}(\lambda)$ enables the computation of the of the correlation function $\mathcal{K}_W(\mathbf{x}) = \mathbf{E}\xi_W(\mathbf{x})\xi^*_W(0)$, i.e.

$$\mathcal{K}_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r}\int\limits_{\mathbf{R}^r} \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} \left(e^{-i\langle 0,\lambda\rangle}\right)_{2W} \mathbf{E}|d\mathbf{Z}(\lambda)|^2 = \int\limits_{\mathbf{R}^r} \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} d\Phi(\lambda),$$

where the isometry (6) is used between $\mathcal{H}(\xi)$ and $L_2(d\Phi; \mathbf{R}^r)$. Thus (16) is proved.

Finally, considering the assumptions of the theorem, from (15) we conclude

$$\mathcal{E}_W(\mathbf{x}) = \sum_{\mathbf{k}:\ k_1,\cdots,k_r\neq 0} |1 - e^{i\langle \mathbf{x}, 2\mathbf{k}W\rangle}|^2\ |\Phi|(C_{r,r}(\mathbf{k})) =$$

$$= 4 \sum_{\mathbf{k}:\ k_1,\cdots,k_r\neq 0} \sin^2\langle \mathbf{x}, \mathbf{k}W\rangle |\Phi|(C_{r,r}(\mathbf{k})).$$

These finish the proof of the theorem. $\Diamond$

4. Final remarks

When only the randomness of the given NBL HRF $\xi(\mathbf{x})$ is considered, then the field $\xi(\mathbf{x})-\xi_W(\mathbf{x})$ is not homogeneous in the general since its second moment, which coincides with the aliasing error $\mathcal{E}_W(\mathbf{x})$, depends on $\mathbf{x}$. Practically, we have $\mathcal{E}_W(x^n) \equiv 0$, $x^n \in \mathcal{L}^{(1)}(W)$; for the onedimensional case (weakly stationary stochastic processes) see Brown(1978).

If $\Phi(\lambda)$ is absolutely continuous with respect to the ordinary Lebesgue-measure $\partial\lambda$, then there exists the spectral density (compare e.g. Yaglom (1987a; 336):

$$\phi(\lambda) = \frac{\partial^r \Phi(\lambda)}{\partial\lambda_1 \cdots \partial\lambda_r}.$$

In this case the assumption $|\Phi|(C_{r,q}(\mathbf{k}), (2\mathbf{k}+1)\underline{W}^{r-q}) = 0$, $q \in \{1, \cdots \ldots, r-1\}$ of the Th. 3. is automatically satisfied. By (16) it follows immediately that

$$\mathcal{E}_W(\mathbf{x}) = \int\limits_{\mathbf{R}^r \setminus \mathcal{W}} \left| e^{i\langle \mathbf{x},\lambda\rangle} - \left(e^{i\langle \mathbf{x},\lambda\rangle}\right)_{2W} \right|^2 \phi(\lambda) d\lambda \le 4 \int\limits_{\mathbf{R}^r \setminus \mathcal{W}} \phi(\lambda) d\lambda. \tag{18}$$

This result is precisely the Brown multidimensional mean-square aliasing error upper bound. Indeed, in Brown (1978)(where just the onedimensional case is studied) it is supposed that there exists the spectral density, because in the Lebesgue-decomposition the possible existence of the singular part of the spectral distribution function can be ignored in applied problems and in practice, see Yaglom (1987b; 2).

The mean-square aliasing error upper bounds (13), (17) and (18) give modest generalizations of the bound (11). However the problem of similar bound for the $|s|$-fold differentiable NBL HRF $\xi(\mathbf{x})$ remains. Also it would be interesting to derive similar aliasing error upper bounds for the harmonizable random fields and for more general NBL HRF as well, when it belongs to the so-called *Lip-classes.* In the latter case we would get some further generalizations of certain results by Habib and Cambanis (1981), derived for non-band-limited harmonizable stochastic processes.

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