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SOLUTIONS OF $||f(\xi+\eta)|| = ||f(\xi)+f(\eta)||$

Peter **Schöpf**

Mathematisches Institut, Universität Graz, Heinrichstraße 36/III, A–8010 Graz, Österreich

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Abstract: The problem of finding or describing all solutions $f:\mathbb{R}\to X$ of $||f(\xi+\eta)|| = ||f(\xi)+f(\eta)||$ had its starting point in a lecture given by Prof. Roman Ger in Graz, where he presented a very special continuous solution by J. Dhombres. This paper presents a characterization of all continuous (and all differentiable) solutions of the functional equation. It generalizes the description in the well-known case, where the norm is strictly convex and therefore all solutions are additive.

Our first task is looking for some necessary conditions which the solutions of the functional equality have to fulfil. In order to formulate this conditions, we need some preparations. We are always working in some real normed vector space $(X, ||\ ||)$ with unitsphere $S := = \{x \in X|\ ||x|| = 1\}$. The convex hull of a set A is denoted by $\operatorname{conv}(A)$ or for short $\operatorname{conv} A$ and $\operatorname{sgn}(\xi)$ is the sign of the real number ξ.

Lemma 1.

$$\underset{0<\lambda<1}{\exists}\ ||x|| = ||(1-\lambda)x+\lambda y|| = ||y|| = 1 \iff \operatorname{conv}\{x,y\}\subset S.$$

Proof. "$\Longleftarrow$": trivial.

"$\Longrightarrow$": Let $z\in \operatorname{int}\operatorname{conv}\{x,y\}$, then there exists a $0<\mu<1$ with $z=(1-\mu)x+\mu y$.

1. *case:* $0 < \mu < \lambda$, then $0 < \lambda - \mu < 1$ and one easily can check, that

$$(1-\lambda)x + \lambda y = \left(1 - \frac{\lambda-\mu}{1-\mu}\right)\left((1-\mu)x + \mu y\right) + \frac{\lambda-\mu}{1-\mu}y$$

and therefore

$$1 = ||(1-\lambda)x + \lambda y|| \leq \left(1 - \frac{\lambda-\mu}{1-\mu}\right)||(1-\mu)x + \mu y|| + \frac{\lambda-\mu}{1-\mu}$$

$$1 \leq ||(1-\mu)x + \mu y||.$$

This together with

$$||(1-\mu)x + \mu y|| \leq (1-\mu)||x|| + \mu||y|| = 1$$

yields $||(1-\mu)x + \mu y|| = 1$ for every $0 < \mu < \lambda$.

2. *case:* $\lambda < \mu < 1$. Same method. $\Diamond$

Lemma 2. *Let $x_1, \ldots, x_n \in X$ nonzero vectors, then*

$$||x_1 + \cdots + x_n|| = ||x_1|| + \cdots + ||x_n|| \iff \operatorname{conv}\left\{\frac{x_1}{||x_1||}, \ldots, \frac{x_n}{||x_n||}\right\} \subset S.$$

Proof. "$\Longleftarrow$": Let $y_i := \frac{x_i}{||x_i||}$, $i = 1, \ldots, n$ and $\lambda_i := \frac{||x_i||}{\sum_{j=1}^{n} ||x_j||}$. $C :=$ $= \operatorname{conv}\{y_1, \ldots, y_n\} \subset S$ implies $\sum_{i=1}^{n} \lambda_i y_i \in C \subset S$, meaning that

$$1 = \left|\left|\sum_{i=1}^{n} \lambda_i \mu_i\right|\right| = \frac{\left|\left|\sum_{i=1}^{n} x_i\right|\right|}{\sum_{i=1}^{n} ||x_i||}.$$

"$\Longrightarrow$": Proof by induction. The case $n = 1$ is trivial. The step from n to $n+1$ is based on our general assumption

$$||x_1 + \cdots + x_n + x_{n+1}|| = ||x_1|| + \cdots + ||x_n|| + ||x_{n+1}||,$$

which implies

$$\begin{aligned} ||x_1 + \cdots + x_n|| &= ||x_1 + \cdots + x_n + x_{n+1} - x_{n+1}|| \geq \\ &\geq ||x_1 + \cdots + x_{n+1}|| - ||x_{n+1}|| = \\ &= ||x_1|| + \cdots + ||x_{n+1}|| - ||x_{n+1}|| = \\ &= ||x_1|| + \cdots + ||x_n||. \end{aligned}$$

Together with $||x_1 + \cdots + x_n|| \leq \sum_{i=1}^{n} ||x_i||$ we get $||\sum_{i=1}^{n} x_1|| = \sum_{i=1}^{n} ||x_i||$. By the same reasoning we get for every nonempty $I \subset \{1, 2, \ldots, n+1\}$

$$(*) \qquad \left\|\sum_{i\in I} x_i\right\| = \sum_{i\in I} ||x_i||.$$

Let $y_i := \frac{x_i}{||x_i||}$, $i = 1,\ldots,n+1$ and $\lambda_i := \frac{||x_i||}{\sum_{j=1}^{n+1} ||x_j||}$, then $p := \sum_{i=1}^{n+1} \lambda_i y_i$ is a convex combination of the $y_1,\ldots,y_{n+1}$ and according to our assumption $||p|| = 1$, i.e. $p \in S$. We will show now, that $\mathrm{conv}\{y_1,\ldots,y_{n+1}\} \subset S$.

$$z \in \mathrm{conv}\{y_1,\ldots,y_{n+1}\} \implies z = \sum_{i=1}^{n+1} \mu_i y_i, \text{ with } \forall_{i=1}^{n+1} \ 0 \le \mu_i \le 1$$

and $\sum_{i=1}^{n+1} \mu_i = 1$. If $\mu_i = 0$ for some i, then there is a nonempty subset $I \subset \{1,\ldots,n+1\}$, $\#I \le n$ and therefore $z \in \mathrm{conv}\{y_i | i \in I\} \subset S$ by $(*)$ and induction hypothesis. Let us therefore assume, that $0 < \mu_i < 1$ for all $i = 1,\ldots,n+1$ and $z \ne p$. Consider the line through p and z:

$$\begin{aligned} y(\tau) := (1-\tau)p + \tau z &= \sum_{i=1}^{n+1} (\lambda_i(1-\tau) + \mu_i\tau)\, y_i = \\ &= \sum_{i=1}^{n+1} (\lambda_i + (\mu_i - \lambda_i)\tau)\, y_i. \end{aligned}$$

Because of $\sum_{i=1}^{n+1} \mu_i = \sum_{i=1}^{n+1} \lambda_i = 1$ and $p \ne z$, there exists an index i_0 with $\mu_{i_0} - \lambda_{i_0} < 0$ and another index i_1 with $\mu_{i_1} - \lambda_{i_1} > 0$. This implies the existence of the following "inf" and "sup":

$$\tau_- := \inf\left\{\tau \in \mathbb{R} \,\middle|\, \forall_{i=1}^{n+1} \ \lambda_i + (\mu_i - \lambda_i)\tau \ge 0\right\},$$

$$\tau_+ := \sup\left\{\tau \in \mathbb{R} \,\middle|\, \forall_{i=1}^{n+1} \ \lambda_i + (\mu_i - \lambda_i)\tau \ge 0\right\}.$$

According to the definition of τ_-, τ_+ we conclude, that $y(\tau_-)$, $y(\tau_+)$ are convex combinations of at most n vectors of the $y_1,\ldots,y_{n+1}$, and therefore by induction hypothesis $y(\tau_-)$, $y(\tau_+) \in S$. But $p = y(0) \in$ $\in S$ is an inner point of $\mathrm{conv}\{y(\tau_-), y(\tau_+)\}$ and hence $z \in \mathrm{conv}\{y(\tau_-), y(\tau_+)\} \subset S$ by Lemma 1. $\diamond$

Lemma 3. *Every solution* $f : \mathbb{R} \to X$ *of* $||f(\xi+\eta)|| = ||f(\xi)+f(\eta)||$ *has the following properties:*

1. $\forall_{\xi\in\mathbb{R}} \; f(-\xi) = -f(\xi);$
2. $\forall_{\kappa\in\mathbb{Q}} \forall_{\xi\in\mathbb{R}} \; ||f(\kappa\xi)|| = |\kappa| \, ||f(\xi)||;$
3. $\forall_{\xi,\eta\in\mathbb{Q},|\xi|\leq|\eta|\neq 0} \; ||f(\xi) + f(\eta)|| = ||f(\eta)|| + \operatorname{sgn}\xi \operatorname{sgn}\eta ||f(\xi)||.$

Proof. Ad 1.

$$||f(0)|| = ||f(0+0)|| = ||f(0) + f(0)|| = \\ = ||2f(0)|| = 2||f(0)|| \Longrightarrow f(0) = o.$$

$$0 = ||f(0)|| = ||f(\xi - \xi)|| = ||f(\xi + (-\xi))|| = \\ = ||f(\xi) + f(-\xi)|| \Longrightarrow f(\xi) = f(-\xi) = o \Longrightarrow f(-\xi) = -f(\xi).$$

Ad 2. 1. step: By induction we show

$$\forall_{\xi\in\mathbb{R}} \forall_{n\in\mathbb{N}} \; ||f(n\xi)|| \leq n||f(\xi)||.$$

The case $n = 1$ is trivial. Now let us do the induction step from n to $n+1$.

$$||f\big((n+1)\xi\big)|| = ||f(n\xi) + f(\xi)|| \leq \\ \leq ||f(n\xi)|| + ||f(\xi)|| \leq n||f(\xi)|| + ||f(\xi)|| = (n+1)||f(\xi)||.$$

2. step: Now we are going to show

$$\forall_{\xi\in\mathbb{R}} \forall_{n\in\mathbb{N}} \; ||f(n\xi)|| = n||f(\xi)||.$$

The case $n = 1$ is trivial. Our induction step requires the case $n = 2$ and therefore, we have to prove it.

$$||f(2\xi)|| = ||f(\xi+\xi)|| = ||f(\xi) + f(\xi)|| = ||2f(\xi)|| = 2||f(\xi)||.$$

Our induction hypothesis is

$$\forall_{1\leq m\leq n} \forall_{\xi\in\mathbb{R}} \; ||f(m\xi)|| = m||f(\xi)||.$$

For the induction step we have to distinguish two cases.

1. *case:* $n+1 = 2k$ with $k \in \mathbb{N}$. This implies $k \leq n$ and our induction hypothesis yields

$$||f\big((n+1)\xi\big)|| = ||f(2k\xi)|| = 2||f(k\xi)|| = 2k||f(\xi)|| = (n+1)||f(\xi)||.$$

2. *case:* $n+1 = 2k+1$ with $k \in \mathbb{N}$. We have $k+1 \leq n$ and with our induction hypothesis we get

$$\begin{aligned}||f((n+1)\xi)|| &= ||f((2k+2)\xi-\xi)|| = ||f(2(k+1)\xi_+(-\xi))|| = \\ &= ||f(2(k+1)\xi)+f(-\xi)|| = ||f(2(k+1)\xi)-f(\xi)|| \geq \\ &\geq ||f(2(k+1)\xi)||-||f(\xi)|| = 2||f((k+1)\xi)||-||f(\xi)|| = \\ &= 2(k+1)||f(\xi)||-||f(\xi)|| = (n+1)||f(\xi)||.\end{aligned}$$

Because of $||f((n+1)\xi)|| \leq (n+1)||f(\xi)||$ we get the desired equality.

3. step: Now let us show

$$\underset{0<\kappa\in\mathbb{Q}}{\forall}\ \underset{\xi\in\mathbb{R}}{\forall}\ ||f(\kappa\xi)|| = \kappa||f(\xi)||.$$

Let $\kappa := \frac{m}{n}$, $m,n \in \mathbb{N}$, then $||f(\frac{m}{n}\xi)|| = m||f(\frac{1}{n}\xi)||$. But $||f(\xi)|| = = ||f(n\frac{1}{n}\xi)|| = n||f(\frac{1}{n}\xi)||$ and therefore $||f(\frac{m}{n}\xi)|| = \frac{m}{n}||f(\xi)||$.

4. step: We show

$$\underset{\kappa\in\mathbb{Q}}{\forall}\ \underset{\xi\in\mathbb{R}}{\forall}\ ||f(\kappa\xi)|| = |\kappa|\,||f(\xi)||.$$

Let $\kappa \in \mathbb{Q}$, then

$$\begin{aligned}||f(\kappa\xi)|| &= ||f(\operatorname{sgn}(\kappa)|\kappa|\xi)|| = ||\operatorname{sgn}(\kappa)f(|\kappa|\xi)|| = \\ &= ||f(|\kappa|\xi)|| = |\kappa|\,||f(\xi)||.\end{aligned}$$

Ad 3. 1. *case:* We show

$$\underset{0\leq\xi,\eta\in\mathbb{Q}}{\forall}\ ||f(\xi)+f(\eta)|| = ||f(\xi)||+||f(\eta)||.$$

$$\begin{aligned}||f(\xi)+f(\eta)|| &= ||f(\xi+\eta)|| = (\xi+\eta)||f(1)|| = \\ &= \xi||f(1)||+\eta||f(1)|| = ||f(\xi)||+||f(\eta)||.\end{aligned}$$

2. *case:* We show

$$\underset{0\leq\xi\leq\eta\in\mathbb{Q}}{\forall}\ ||f(\xi)-f(\eta)|| = ||f(\xi)||-||f(\eta)||.$$

$$\begin{aligned}||f(\eta)-f(\xi)|| &= ||f(\eta-\xi)|| = (\eta-\xi)||f(1)|| = \\ &= \eta||f(1)||-\xi||f(1)|| = ||f(\eta)||-||f(\xi)||.\end{aligned}$$

3. *case:* Let $\xi,\eta \in \mathbb{Q}$ and $|\xi| \leq |\eta|$. We show

$$||f(\eta)+f(\xi)|| = ||f(\eta)||+\operatorname{sgn}\xi\operatorname{sgn}\eta||f(\xi)||.$$

Without loss of generality $\xi \neq 0 \neq \eta$;

$$\begin{aligned}||f(\eta)+f(\xi)|| &= ||f(\operatorname{sgn}\eta|\eta|)+f(\operatorname{sgn}\xi|\xi|)|| = \\ &= ||\operatorname{sgn}\eta f(|\eta|)+\operatorname{sgn}\xi f(|\xi|)|| = \left\| f(|\eta|)+\frac{\operatorname{sgn}\xi}{\operatorname{sgn}\eta} f(|\xi|)\right\| = \\ &= ||f(|\eta|)+\operatorname{sgn}\xi \operatorname{sgn}\eta||f(|\xi|) = \\ &= ||f(\eta)||+\operatorname{sgn}\xi \operatorname{sgn}\eta||f(\xi)||. \quad \diamond\end{aligned}$$

Theorem 1. *Every continuous solution: $f:\mathbb{R}\to X$ of $||f(\xi+\eta)|| = ||f(\xi)+f(\eta)||$ has the following properties.*

1. $\forall_{\xi\in\mathbb{R}}\ f(-\xi)=-f(\xi)$;
2. $\forall_{\xi,\eta\in\mathbb{R}}\ ||f(\xi\eta)|| = |\xi|\,||f(\eta)||$;
3. $\forall_{\xi,\eta\in\mathbb{R},|\xi|\le|\eta|}\ ||f(\xi)+f(\eta)|| = ||f(\eta)||+\operatorname{sgn}\xi \operatorname{sgn}\eta||f(\xi)||$;
4. *if $f\neq 0$, then* $\operatorname{conv}\left\{\dfrac{f(\eta)-f(\xi)}{||f(\eta)-f(\xi)||}\Big|\xi<\eta;\ \xi,\eta\in\mathbb{R}\right\}\subset S$.

Proof. Ad 1–3, Lemma 3 and continuity of f.

Ad 4. Let us assume that $f\neq 0$. By 1. and 2. we infer, that f is injective and therefore $f(\eta)-f(\xi)\neq 0$ if $\xi<\eta$. We must show, that the convex hull of finitely many vectors $\frac{f(\eta_i)-f(\xi_i)}{||f(\eta_i)-f(\xi_i)||}$, $\xi_i<\eta_i$, $i=1,\dots,n$ is contained in S. For this task consider the set of arguments $A:=\{\xi_i,\eta_i|i=1,\dots,n\}=\{\alpha_1,\dots,\alpha_m\}$ with $\alpha_1<\alpha_2<\dots<\alpha_m$, if all $\xi_i\ge 0$ or if all $\eta_i\le 0$. If there are indices i and j with $\xi_i<0$ and $\eta_j>0$, then $A:=\{\xi_i,\eta_i|i=1,\dots,n\}\cup\{0\}$, i.e. $\alpha_1<\alpha_2\dots<\alpha_l=0<\dots<\alpha_m$. Let $\eta_i=\alpha_{k(i)}$ and $\xi_i=\alpha_{l(i)}$ then we can write

$$\eta_i-\xi_i=\alpha_{k(i)}-\alpha_{l(i)}=\sum_{j=l(i)}^{k(i)-1}\left(\alpha_{j+1}-\alpha_j\right).$$

1. step: We want to show, that

$$D:=\operatorname{conv}\left\{\frac{f(\alpha_{j+1})-f(\alpha_j)}{||f(\alpha_{j+1})-f(\alpha_j)||}\Big|j=1,\dots,m\right\}\subset S.$$

By Lemma 2, we only have to show, that

$$\left\|\sum_{j=1}^{m-1}\left(f(\alpha_{j+1})-f(\alpha_j)\right)\right\| = \sum_{j=1}^{m-1}||(f(\alpha_{j+1})-f(\alpha_j))||.$$

This is done by the following formal manipulations with the aid of Th. 1.3.

$$\left\|\sum_{j=1}^{m-1}\left(f(\alpha_{j+1})-f(\alpha_j)\right)\right\| = ||f(\alpha_m)-f(\alpha_1)|| =$$

$$= \begin{cases} ||f(\alpha_m)||-||f(\alpha_1)||, & \text{if } 0\le \alpha_1<\alpha_m \\ ||f(\alpha_m)||+||f(\alpha_1)||, & \text{if } \alpha_1<0<\alpha_m \\ ||f(\alpha_1)||-||f(\alpha_m)||, & \text{if } \alpha_1<\alpha_m\le 0 \end{cases}$$

$$= \begin{cases} \sum_{j=1}^{m-1}(||f(\alpha_{j+1})||-||f(\alpha_j)||) \\ \sum_{j=1}^{l-1}(||f(\alpha_j)||-||f(\alpha_{j+1})||)+\sum_{j=l}^{m-1}(||f(\alpha_{j+1})||-||f(\alpha_j)||), \\ \quad \text{remember } \alpha_l=0 \text{ and } f(0)=o \\ \sum_{j=1}^{m-1}(||f(\alpha_j)||-||f(\alpha_{j+1})||) \end{cases}$$

$$= \begin{cases} \sum_{j=1}^{m-1}||f(\alpha_{j+1})||-||f(\alpha_j)|| \\ \sum_{j=1}^{l-1}||f(\alpha_j)-f(\alpha_{j+1})||)+\sum_{j=l}^{m-1}||f(\alpha_{j+1})-f(\alpha_j)||, \\ \sum_{j=1}^{m-1}||f(\alpha_j)-f(\alpha_{j+1})|| \end{cases}$$

$$= \sum_{j=1}^{m-1}||f(\alpha_{j+1})-f(\alpha_j)||$$

hence $D\subset S$.

2. step: Formula $(*)$ of Lemma 2 allows the conclusion

$$(**)\qquad \left\|\sum_{j=l(i)}^{k(i)-1}\left(f(\alpha_{j+1})-f(\alpha_j)\right)\right\| = \sum_{j=l(i)}^{k(i)-1}||f(\alpha_{j+1})-f(\alpha_j)||$$

and hence (again by Lemma 2)

$$\underset{i=1}{\overset{n}{\forall}} \; D_i := \operatorname{conv}\left\{\frac{f(\alpha_{j+1})-f(\alpha_j)}{||f(\alpha_{j+1})-f(\alpha_j)||}\middle| l(i)\le j\le k(i)\right\}\subset S.$$

With (**) we get

$$\frac{f(\eta_i)-f(\xi_i)}{||f(\eta_i)-f(\xi_i)||}=\frac{\sum\limits_{j=l(i)}^{k(i)-1}\left(f(\alpha_{j+1})-f(\alpha_j)\right)}{\left\|\sum\limits_{j=l(i)}^{k(i)-1}\left(f(\alpha_{j+1})-f(\alpha_j)\right)\right\|}=$$

$$=\sum_{j=l(i)}^{k(i)-1}\frac{||f(\alpha_{j+1})-f(\alpha_j)||}{\sum\limits_{j=l(i)}^{k(i)-1}||f(\alpha_{j+1})-f(\alpha_j)||}\;\frac{f(\alpha_{j+1})-f(\alpha_j)}{||f(\alpha_{j+1})-f(\alpha_j)||}\in D_i.$$

Because of

$$\underset{i=1}{\overset{n}{\forall}} \; \frac{f(\eta_i)-f(\xi_i)}{||f(\eta_i)-f(\xi_i)||}\in D_i\subset D\subset S$$

and the convexity of D, we conclude

$$\operatorname{conv}\left\{\frac{f(\eta_i)-f(\xi_i)}{||f(\eta_i)-f(\xi_i)||}\middle| i=1,\dots,n\right\}\subset S. \qquad \diamondsuit$$

Corollary 1. *Let $f:\mathbb{R}\to X$ be a solution of $||f(\xi+\eta)||=||f(\xi)+$ $+f(\eta)||$ and assume, that $\frac{f(\xi_0)}{||f(\xi_0)||}$ is an exposed point of S (i.e. the maximal convex subset C of S containing this point is $\left\{\frac{f(\xi_0)}{||f(\xi_0)||}\right\}$) for some $\xi_0\in\mathbb{R}\setminus\{0\}$. Then*

$$\underset{\xi,\eta\in\mathbb{R}}{\forall} \; f(\xi+\eta)=f(\xi)+f(\eta).$$

Proof. Without loss of generality we may assume $\xi_0>0$. Th. 1 (4) then says, that $\operatorname{conv}\left\{\frac{f(\xi)-f(0)}{||f(\xi)-f(0)||}\middle|\xi>0\right\}\subset S$ and therefore $\frac{f(\xi)}{||f(\xi)||}=$ $=\frac{f(1)}{||f(1)||}=\frac{f(\xi_0)}{||f(\xi_0)||}$ for every $\xi>0$, i.e.

$$\underset{\xi>0}{\forall} \; f(\xi)=||f(\xi)||\frac{f(1)}{||f(1)||}=\xi||f(1)||\,\frac{f(1)}{||f(1)||}=\xi f(1).$$

But by Lemma 3 (1) we get

$$\underset{\xi\in\mathbb{R}}{\forall} \; f(\xi)=\xi f(1),$$

which implies additivity of f. $\diamondsuit$

Theorem 2. *Let $f : \mathbb{R} \to X$ a function satisfying the following conditions*

1. $\forall_{\xi\in\mathbb{R}}\ f(-\xi) = -f(\xi)$,
2. $\forall_{\xi,\eta\in\mathbb{R}}\ ||f(\xi\eta)|| = |\xi|\,||f(\eta)||$,
3. $\forall_{\xi_1<\eta_1\in\mathbb{R}}\ \forall_{\xi_2<\eta_2\in\mathbb{R}}\ \mathrm{conv}\left\{\frac{f(\eta_1)-f(\xi_1)}{||f(\eta_1)-f(\xi_1)||}, \frac{f(\eta_2)-f(\xi_2)}{||f(\eta_2)-f(\xi_2)||}\right\} \subset S.$

Then f is continuous and

$$\forall_{\xi,\eta\in\mathbb{R}}\ ||f(\xi+\eta)|| = ||f(\xi)+f(\eta)||.$$

Remark. Condition 3 in Th. 2 is weaker than condition 4 in Th. 1.
Proof. The third condition can reformulated (according Lemma 2) as follows
$||f(\eta_2)-f(\xi_2)+f(\eta_1)-f(\xi_1)|| = ||f(\eta_2)-f(\xi_2)|| + ||f(\eta_1)-f(\xi_1)||$.

Now let $\xi, \eta \in \mathbb{R}\setminus\{0\}$, $\xi < \eta$, then we have to distinguish four cases.

1. *case:* $0 < \xi < \eta$. If we define $\xi_1 = \xi_2 = 0 < \eta_1 = \xi < \eta_2 = \eta$, then we get by 2 and 3.

$$||f(\eta)+f(\xi)|| = ||f(\eta)|| + ||f(\xi)|| =$$
$$\eta||f(1)|| + \xi||f(1)|| = (\eta+\xi)||f(1)|| = ||f(\eta+\xi)||.$$

2. *case:* $0 < -\xi \le \eta$. If we define $\xi_1 = 0 < \eta_1 = \xi_2 = -\xi < \eta_2 = \eta$, then we get by 1, 2 and 3.
$||f(\eta)|| = ||f(\eta)-f(-\xi)+f(-\xi)-f(0)|| = ||f(\eta)-f(-\xi)|| + ||f(-\xi)||$

$$||f(\eta)+f(\xi)|| = ||f(\eta)|| - ||f(\xi)|| = \eta||f(1)|| - (-\xi)||f(1)|| =$$
$$= (\eta+\xi)||f(1)|| = ||f(\eta+\xi)||.$$

3. *case:* $0 < \eta \le -\xi$. If we define $\xi_1 = 0 < \eta_1 = \xi_2 = \eta < \eta_2 = -\xi$ then we can use the second case to get

$$||f(\eta)+f(\xi)|| = ||f(\eta+\xi)||.$$

4. *case:* $\xi < \eta < 0$. Because of $0 < -\eta < -\xi$ we can use the first case to get the desired functional relation.

To see continuity of f, we only have to lock into the following line.

$$||f(\xi+\delta)-f(\xi)|| = ||f(\delta)|| = |\delta|\,||f(1)|| \underset{\delta\to 0}{\longrightarrow} 0. \quad \Diamond$$

Theorem 3. *Let $(X, \|\ \|)$ be a Banach space. Every nonzero differentiable solution $f : \mathbb{R} \longrightarrow X$ of $\|f(\xi+\eta)\| = \|f(\xi)+f(\eta)\|$ has beside all properties stated in* Th. 1 *for continuous solutions the following properties*

1. $\underset{\xi\in\mathbb{R}}{\forall}\ f'(-\xi) = f'(\xi)$,
2. $\underset{\xi\in\mathbb{R}}{\forall}\ \|f'(\xi)\| = \|f(1)\| > 0$,
3. $\operatorname{conv}\left\{\frac{f'(\xi)}{\|f(1)\|}\Big|\xi\in\mathbb{R}\right\} \subset \operatorname{cl}\left(\operatorname{conv}\left\{\frac{f(\eta)-f(\xi)}{\|f(\eta)-f(\xi)\|}\Big|\xi<\eta\right\}\right) \subset S.$

Proof. Ad 1. Th. 1 (1) says $f(-\xi) = -f(\xi)$ and therefore $f'(-\xi) = = f'(\xi)$.

Ad 2. Let $\xi, \delta \in \mathbb{R}$ and $\delta \neq 0$, then

$$\|f(\xi+\delta) - f(\xi)\| = \|f(\delta)\| = |\delta|\,\|f(1)\|$$

and therefore $\|f'(\xi)\| = \|f(1)\|$, with $\|f(1)\| > 0$ because f is a nonconstant function.

Ad 3. It is easy to see, that

$$\frac{\dfrac{f(\xi+\delta)-f(\xi)}{\delta}}{\left\|\dfrac{f(\xi+\delta)-f(\xi)}{\delta}\right\|} = \begin{cases} \dfrac{f(\xi+\delta)-f(\xi)}{\|f(\xi+\delta)-f(\xi)\|}, & \text{if } \delta > 0 \\ \dfrac{f(\xi)-f(\xi-|\delta|)}{\|f(\xi)-f(\xi-|\delta|)\|}, & \text{if } \delta < 0 \end{cases}$$

and therefore by Th. 3 (2) and Th. 1 (4).

$$\frac{f'(\xi)}{\|f(1)\|} = \frac{f'(\xi)}{\|f'(\xi)\|} = \lim_{\delta\to 0} \frac{\dfrac{f(\xi+\delta)-f(\xi)}{\delta}}{\left\|\dfrac{f(\xi+\delta)-f(\xi)}{\delta}\right\|} \in \operatorname{cl}(C) \subset S$$

where $\operatorname{cl}(C)$ is the topological closure of the convex set $C \subset S$ from Th. 1(4) $\operatorname{cl}(C)$ is convex and contained in S because S is a closed set in X. Therefore we have $\operatorname{conv}\left\{\frac{f'(\xi)}{\|f(1)\|}\Big|\xi\in\mathbb{R}\right\} \subset S$. ◊

Theorem 4. *Let $(X, \|\ \|)$ be a Banach space and $f : \mathbb{R} \to X$, $f(0) = 0$, $f(1) \neq 0$ a differentiable function satisfying*

1. $\underset{\xi\in\mathbb{R}}{\forall}\ f'(-\xi) = f'(\xi)$ *and* 2. $\operatorname{conv}\left\{\frac{f'(\xi)}{\|f(1)\|}\Big|\xi\in\mathbb{R}\right\} \subset S.$

Then f is a solution of $\|f(\xi+\eta)\| = \|f(\xi)+f(\eta)\|$.

Proof. We only have to show, that f fulfills three conditions of Th. 1. The function $\xi \mapsto f(-\xi) + f(\xi)$ is differentiable and therefore we get

$$||f(-\xi)+f(\xi)-2f(0)|| \le$$
$$\le |\xi - 0| \sup\{||-f'(-\tau)+f'(\tau)|| \, | 0 \le \tau \le \xi\} = 0$$

i.e. $f(-\xi) = -f(\xi)$. From 2. we get $||f'(\xi)|| = ||f(1)||$ and hence $||f(\xi)-f(0)|| = |\xi - 0| \, ||f(1)||$, i.e. $||f(\xi)|| = |\xi| \, ||f(1)||$. The following line

$$||f(\eta\xi)|| = |\eta| \, |\xi| \, ||f(1)|| = |\xi| \, ||f(\eta)||$$

shows, that Th. 2 (2) is fulfilled. Let D be the convex set in 2, then we can infer, that

$$\forall_{\xi<\eta} \; f(\eta)-f(\xi) \in ||f(\eta)-f(\xi)|| \, \mathrm{cl}\, D \subset S.$$

Since $\mathrm{cl}\, D$ is convex, we see, that Th. 2 (3) is satisfied. ◊

Remark. We also have discontinuous solutions. If $\alpha : \mathbb{R} \to \mathbb{R}$ is a discontinuous solution of the Cauchy equation $\alpha(\xi+\eta) = \alpha(\xi)+\alpha(\eta)$ and $f : \mathbb{R} \to X$ is a continuous solution of $||f(\xi+\eta)|| = ||f(\xi) + f(\eta)||$ with $f(1) \neq 0$, then $f \circ \alpha$ is a discontinuous solution of our functional equation, because $||f(\alpha(\xi))|| = |\alpha(\xi)| \, ||f(1)||$. It seems to be an open problem, whether every discontinuous solution of our functional equation has this special form.

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