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# UNIFORMLY, PROXIMALLY AND TOPOLOGICALLY COMPACT RELATORS

Árpád **Száz**

*Institute of Mathematics and Informatics, Kossuth Lajos University, H-4010 Debrecen, Pf. 12, Hungary*

**Dedicated to the memory of Jenő Deák**

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**Abstract:** By using relator spaces instead of topological and uniform spaces, we extend, unify and supplement several standard characterization theorems on compactness and precompactness.

## Introduction

In this paper, we show that the following basic characterization theorems of Kelley [11, p. 136], Gál [9] and Sieber and Pervin [20] can be nicely extended to relator spaces [21], [24], which are, in a certain sence, the ultimate reasonable generalizations of Weil's uniform spaces [30].

**Theorem 1.** *A topological space $X$ is compact if and only if every family of closed subsets of $X$ having the finite intersection property has a non-void intersection.*

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**Theorem 2.** *A topological space $X$ is compact if and only if every net in $X$ has a convergent subnet.*

**Theorem 3.** *A regular topological space $X$ is compact if it has a dense subset $A$ such that every net in $A$ has a subnet which converges in $X$.*

**Theorem 4.** *A quasi-uniform space $X$ is precompact if and only if every net in $X$ has a Cauchy subnet.*

**Theorem 5.** *A quasi-uniform space $X$ is compact if and only if it is both complete and precompact.*

**Theorem 6.** *A topological space $X$ is compact if and only if it is complete with respect to every quasi-uniformity which generates its topology.*

More precisely, taking up an unnoticed idea of Konishi [12, p. 169] and Nakano–Nakano [17, p. 211] that compactness is actually a particular case of precompacness, we show that Th. 4 has a far reaching generalization which easily yields several extensions of the above theorems.

## 1. A few basic facts on relators

If $\mathcal{R}$ is a family of binary relations on a set $X$, then we say that the family $\mathcal{R}$ is a *relator* on $X$ and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is a relator space. (See [21] for the origins.)

If $(x_\alpha)$ is a net in $X(\mathcal{R})$, then $\lim_{\mathcal{R}}(x_\alpha)$ $\big(\mathrm{adh}_{\mathcal{R}}(x_\alpha)\big)$ denotes the set of all points $x$ in $X$ such that $(x_\alpha)$ is eventually (frequently) in $R(x)$ for all $R \in \mathcal{R}$. (See [11, p. 65].)

The net $(x_\alpha)$ is called *convergent (adherent)* if $\lim_{\mathcal{R}}(x_\alpha) \neq \emptyset$ $\big(\mathrm{adh}_{\mathcal{R}}(x_\alpha) \neq \emptyset\big)$. Since $\alpha$ can now be required to run only in a nonvoid preordered set, a convergent net need not be adherent.

If $A$ is a set in $X(\mathcal{R})$, then $A^- = \mathrm{cl}_{\mathcal{R}}(A)$ $\big(A^\circ = \mathrm{int}_{\mathcal{R}}(A)\big)$ denotes the set of all points $x$ in $X$ such that $R(x) \cap A \neq \emptyset$ $\big(R(x) \subset A\big)$ for all (some) $R \in \mathcal{R}$.

The set $A$ is called *closed (open)* if $A^- \subset A$ $\big(A \subset A^\circ\big)$. Since now the set $A^-$ $(A^\circ)$ need not be closed (open), the closed (open) sets cannot play an important role here.

If $\mathcal{R}$ is a relator on $X$, then the relator $\mathcal{R}^{-1} = \big\{R^{-1} : R \in \mathcal{R}\big\}$ is called the *inverse* of $\mathcal{R}$, and the relators

$$\mathcal{R}^* = \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\},$$
$$\mathcal{R}^\# = \{S \subset X^2 : \forall A \subset X : \exists R \in \mathcal{R} : R(A) \subset S(A)\},$$
$$\mathcal{R}^\wedge = \{S \subset X^2 : \forall x \in X : \exists R \in \mathcal{R} : R(x) \subset S(x)\}$$

are called the *uniform, proximal* and *topological refinements* of $\mathcal{R}$, respectively.

The latter terminology is mainly motivated by the fact that if $\mathcal{R} \neq \neq \emptyset$, then $\mathcal{R}^\wedge$ is the largest relator on $X$ such that $\lim_{\mathcal{R}^\wedge} = \lim_{\mathcal{R}} \big(\operatorname{adh}_{\mathcal{R}^\wedge} = = \operatorname{adh}_{\mathcal{R}}\big)$ or $\operatorname{cl}_{\mathcal{R}^\wedge} = \operatorname{cl}_{\mathcal{R}}$ $\big(\operatorname{int}_{\mathcal{R}^\wedge} = \operatorname{int}_{\mathcal{R}}\big)$.

Two relators $\mathcal{R}$ and $\mathcal{S}$ on $X$ are called *topologically (proximally) equivalent* if $\mathcal{R}^\wedge = \mathcal{S}^\wedge$ ($\mathcal{R}^\# = \mathcal{S}^\#$). Therefore, relators generating the same closed (open) sets need not be topologically equivalent.

If $\mathcal{R}$ is a relator on $X$, then we say that:

(1) $\mathcal{R}$ is *non-degenerated* if $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$; $\mathcal{R}$ is *non-partial* if $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$.

(2) $\mathcal{R}$ is *reflexive* if each $R \in \mathcal{R}$ is reflexive; $\mathcal{R}$ is *quasi-topological* if $x \in R(x)^{\circ\circ}$ for all $x \in X$ and $R \in \mathcal{R}$; $\mathcal{R}$ is *topological* if $\mathcal{R}$ is reflexive and quasi-topological.

(3) $\mathcal{R}$ is *topologically symmetric* if for each $x \in X$ and $R \in \mathcal{R}$ there exists an $S \in \mathcal{R}$ such that $S(x) \subset R^{-1}(x)$; $\mathcal{R}$ is *topologically transitive* if for each $x \in X$ and $R \in \mathcal{R}$ there exist $S, T \in \mathcal{R}$ such that $T(S(x)) \subset R(x)$.

(4) $\mathcal{R}$ is *strongly symmetric* if each $R \in \mathcal{R}$ is symmetric; $\mathcal{R}$ is *uniformly filtered* if for each $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subset R \cap S$.

Whenever a relator $\mathcal{R}$ on $X$ has a property $P$, then we also say that the relator space $X(\mathcal{R})$ *has the property* $P$. Moreover, if the inverse relator $\mathcal{R}^{-1}$ is $P$, then we also say that $\mathcal{R}$ is *inversely* $P$.

Quite similarly, a subset $A$ of a relator space $X(\mathcal{R})$ is called *inversely dense* if $\operatorname{cl}_{\mathcal{R}^{-1}}(A) = X$. Therefore, a dense subset of a topologically symmetric relator space is inversely dense.

Finally, we remark that a family $\mathcal{A}$ of subsets of a set $X$ is called *centred* [3, p. 57] if the family $\mathcal{A}'$ of all finite intersections of members of $\mathcal{A}$ does not contain the empty set.

A family $\mathcal{A}$ of subset of a relator space $X(\mathcal{R})$ is called an *interior cover* [1, p. 285] of $X(\mathcal{R})$ if the family $\mathcal{A}^\circ = \{A^\circ : A \in \mathcal{A}\}$ is a cover of $X$ in the sense that $\bigcup \mathcal{A}^\circ = X$.

## 2. Cauchy nets and a mixed completeness

**Definition 2.1.** A net $(x_\alpha)$ in a relator space $X(\mathcal{R})$ is called *convergence (adherence) Cauchy* if it is convergent (adherent) in each of the spaces $X(\{R\})$, where $R \in \mathcal{R}$. Moreover, $(x_\alpha)$ is called *uniformly, proximally* and *topologically convergence (adherence) Cauchy* if it is convergence (adherence) Cauchy in the spaces $X(\mathcal{R}^*)$, $X(\mathcal{R}^\#)$ and $X(\mathcal{R}^\wedge)$, respectively.

The appropriateness of this unusual definition and the validity of the next important theorem have been partly established in [26].

**Theorem 2.2.** *If $(x_\alpha)$ is a net in a relator space $X(\mathcal{R})$, then the following assertions are equivalent:*

(1) *$(x_\alpha)$ is convergent (adherent);*
(2) *$(x_\alpha)$ is topologically convergence (adherence) Cauchy.*

**Remark 2.3.** Hence, it is also clear that a convergent (adherent) net is, in particular, convergence (adherence) Cauchy.

**Definition 2.4.** A relator $\mathcal{R}$ on $X$ is called *directedly convergence-adherence complete* if each directed convergence Cauchy net in $X(\mathcal{R})$ is adherent. Moreover, $\mathcal{R}$ is called *uniformly, proximally* and *topologically directedly convergence-adherence complete* if the relators $\mathcal{R}^*$, $\mathcal{R}^\#$ and $\mathcal{R}^\wedge$ are directedly convergence-adherence complete, respectively.

The appropriateness of this particular definition and the validity of the next useful theorem have also been established in [26].

**Theorem 2.5.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is directedly convergence-adherence complete;*
(2) *each directed universal convergence Cauchy net in $X(\mathcal{R})$ is convergent.*

**Remark 2.6.** Note that a net may naturally be called universal if it is eventually in every set in which it is frequently.

## 3. Uniform, proximal and topological compactnesses

**Definition 3.1.** A relator $\mathcal{R}$ on $X$ will be called *compact* if for each $R$ in $\mathcal{R}$ there exists a finite subset $A$ of $X$ such that $R(A) = X$. Moreover, $\mathcal{R}$ will be called *uniformly, proximally* and *topologically compact* if the relators $\mathcal{R}^*$, $\mathcal{R}^\#$ and $\mathcal{R}^\wedge$ are compact, respectively.

**Remark 3.2.** Because of the inclusions $\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\# \subset \mathcal{R}^\wedge$, it is clear that "topologically compact" $\Longrightarrow$ "proximally compact" $\Longrightarrow$ $\Longrightarrow$ "uniformly compact" $\Longrightarrow$ "compact". Moreover, using the corresponding definitions, one can easily see that "compact" also implies "uniformly compact", and thus these latter terms are equivalent. On the other hand, the next useful constructions show that in general "uniformly compact" $\not\Longrightarrow$ "proximally compact" $\not\Longrightarrow$ "topologically compact".

**Theorem 3.3.** *If $\mathcal{R}$ is an inversely non-partial relator on a nonvoid set $X$, and for each $x \in X$ and $R \in \mathcal{R}$ we set $V_{(x,R)} \subset X \times X$ such that* $V_{(x,R)}(z) = X$ *if* $z = x$ *and* $V_{(x,R)}(z) = R(z)$ *if* $z \in X \setminus \{x\}$, *then*

$$\mathcal{V}_\mathcal{R} = \{V_{(x,R)} : x \in X,\ R \in \mathcal{R}\}$$

*is a uniformly compact relator on $X$ such that $\mathcal{R}$ and $\mathcal{V}_\mathcal{R}$ are proximally equivalent.*

**Proof.** Since $V_{(x,R)}(\{x\}) = X$ for all $x \in X$ and $R \in \mathcal{R}$, it is clear that $\mathcal{V}_\mathcal{R}$ is, in particular, compact, and hence it is also uniformly compact. Moreover, since $R \subset V_{(x,R)}$ for all $x \in X$ and $R \in \mathcal{R}$, it is clear that in particular we have $\mathcal{V}_\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\#$, and hence $(\mathcal{V}_\mathcal{R})^\# \subset \mathcal{R}^\#$.

On the other hand, if $R \in \mathcal{R}$, and $A$ is a proper subset of $X$ and $x \in X \setminus A$, then it is clear that

$$V_{(x,R)}(A) = \bigcup_{a \in A} V_{(x,R)}(a) = \bigcup_{a \in A} R(a) = R(A).$$

Moreover, if $R \in \mathcal{R}$ and $x \in X$, then since $\mathcal{R}$ is inversely non-partial it is clear that

$$V_{(x,R)}(X) = X = R(X).$$

Therefore, $\mathcal{R} \subset (\mathcal{V}_\mathcal{R})^\#$, and hence $\mathcal{R}^\# \subset (\mathcal{V}_\mathcal{R})^\#$ is also true. Consequently, $\mathcal{R}^\# = (\mathcal{V}_\mathcal{R})^\#$. $\Diamond$

As an immediate consequence of this theorem, now we can also state

**Example 3.4.** If $X$ is an infinite set and $\mathcal{R} = \{\Delta_X\}$, then $\mathcal{V}_\mathcal{R}$ is a uniformly compact relator on $X$ such that $\mathcal{V}_\mathcal{R}$ is not proximally compact. Namely, we now have $\Delta_X \in \mathcal{R} \subset \mathcal{R}^\# = (\mathcal{V}_\mathcal{R})^\#$, and thus $(\mathcal{V}_\mathcal{R})^\#$ cannot be compact.

**Remark 3.5.** Note that in this case the relators $(\mathcal{V}_\mathcal{R})'$ and $(\mathcal{V}_\mathcal{R})^{-1}$ are not compact either.

Analogously to Th. 3.3, we can also prove the following more important

**Theorem 3.6.** *If $\mathcal{R}$ is a relator on $X$, with* card$(X) > 1$, *and for each $x \in X$ and $R \in \mathcal{R}$ we set $W_{(x,R)} \subset X \times X$ such that*
$W_{(x,R)}(z) = R(x)$ *if* $z = x$ *and* $W_{(x,R)}(z) = X$ *if* $z \in X \setminus \{x\}$, *then*

$$\mathcal{W}_{\mathcal{R}} = \{W_{(x,R)} : x \in X,\ R \in \mathcal{R}\}$$

*is a proximally compact relator on $X$ such that $\mathcal{R}$ and $\mathcal{V}_{\mathcal{R}}$ are topologically equivalent.*

**Proof.** If $a, b \in X$, with $a \neq b$, and $A = \{a, b\}$, then we evidently have

$$W_{(x,R)}(A) = W_{(x,R)}(a) \cup W_{(x,R)}(b) = X$$

for all $x \in X$ and $R \in \mathcal{R}$. And hence, it is clear that we also have $W(A) = X$ for all $W \in (\mathcal{W}_{\mathcal{R}})^{\#}$. Thus, $\mathcal{W}_{\mathcal{R}}$ is, in particular, proximally compact.

On the other hand, since

$$R \subset W_{(x,R)} \quad \text{and} \quad W_{(x,R)}(x) = R(x)$$

for all $x \in X$ and $R \in \mathcal{R}$, it is clear that in particular we also have

$$\mathcal{W}_{\mathcal{R}} \subset \mathcal{R}^* \subset \mathcal{R}^\wedge \quad \text{and} \quad \mathcal{R} \subset (\mathcal{W}_{\mathcal{R}})^\wedge,$$

and hence $\mathcal{R}^\wedge = (\mathcal{W}_{\mathcal{R}})^\wedge$. $\Diamond$

As an immediate consequence of this theorem, now we can also state

**Example 3.7.** If $X$ is an infinite set and $\mathcal{R} = \{\Delta_X\}$, then $\mathcal{W}_{\mathcal{R}}$ is a proximally compact relator on $X$ such that $\mathcal{W}_{\mathcal{R}}$ is not topologically compact. Namely, we now have $\Delta_X \in \mathcal{R} \subset \mathcal{R}^\wedge = (\mathcal{W}_{\mathcal{R}})^\wedge$, and thus $(\mathcal{W}_{\mathcal{R}})^\wedge$ cannot be compact.

**Remark 3.8.** Note that in this case the relators $(\mathcal{W}_{\mathcal{R}})'$ and $(\mathcal{W}_{\mathcal{R}})^{-1}$ are also compact.

## 4. Generalized sequential characterizations of compactnesses

**Theorem 4.1.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is compact;*
(2) *each directed net in $X(\mathcal{R})$ is adherence Cauchy;*
(3) *each directed universal net in $X(\mathcal{R})$ is convergence Cauchy;*
(4) *each directed net in $X(\mathcal{R})$ has a directed convergence Cauchy subnet;*
(5) *each directed net in $X(\mathcal{R})$ has a directed adherence Cauchy subnet.*

(6) *each directed net in $X(\mathcal{R})$ has an adherence Cauchy subnet.*

**Proof.** If (2) does not hold, then there exist a directed net $(x_\alpha)$ in $X(\mathcal{R})$ and an $R \in \mathcal{R}$ such that $(x_\alpha)$ is eventually in $X \setminus R(x)$ for all $x \in X$. On the other hand, if (1) holds, then there exists a finite subset $A$ of $X$ such that $X = R(A) = \bigcup_{a \in A} R(a)$, and hence $\bigcap_{a \in A}(X \setminus R(a)) = \emptyset$. Hence, because of the directedness of $(x_\alpha)$, it follows that $(x_\alpha)$ is eventually in the empty set, which is a contradiction. Therefore, (1) implies (2).

The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are immediate consequences of the facts that

(a) each adherent universal net is convergent;

(b) each directed net has a directed universal subnet [10, Th. 20];

(c) each directed convergent net is adherent. Moreover, (5) trivially implies (6).

Therefore, we need only show that (6) also implies (1). For this note that if (1) does not hold, then there exists an $R \in \mathcal{R}$ such that for each finite subset $A$ of $X$ there exists an $x_A \in X$ such that $x_A \notin R(A)$. Hence, by directing the family of all finite subsets of $X$ with the ordinary set inclusion, we can at once state that $(x_A)$ is a directed net in $X(\mathcal{R})$ such that $(x_A)$ has no adherence Cauchy subnet. And thus (6) cannot hold. Namely, if $(x_{A_\alpha})$ is an adherence Cauchy subnet of $(x_A)$, then there exists an $x \in X$ such that $(x_{A_\alpha})$ is frequently in $R(x)$. Moreover, there exists an $\alpha_0$ such that $\{x\} \subset A_\alpha$ for all $\alpha \geq \alpha_0$. Hence, by choosing $\alpha_1 \geq \alpha_0$ such that $x_{A_{\alpha_1}} \in R(x)$, we clearly have $x_{A_{\alpha_1}} \in R(A_{\alpha_1})$, which contradicts the choice of $(x_A)$. $\Diamond$

**Remark 4.2.** Note that the standard sequential proof of Kelley [11, p. 199] cannot be applied in the present generality.

From Th. 4.1, by Th. 2.2, it is clear that we also have

**Theorem 4.3.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically compact;*

(2) *each directed net in $X(\mathcal{R})$ is adherent;*

(3) *each directed universal net in $X(\mathcal{R})$ is convergent;*

(4) *each directed net in $X(\mathcal{R})$ has a directed convergent subnet;*

(5) *each directed net in $X(\mathcal{R})$ has a directed adherent subnet;*

(6) *each directed net in $X(\mathcal{R})$ has an adherent subnet.*

**Remark 4.4.** Note that traditionally the corresponding particular cases of Th. 4.3 are considered to be completely independent from those of

Th. 4.1. (See, for instance, [11, pp. 136].)

From Th. 4.3 and 4.1, by Th. 2.5, it is clear that we also have

**Theorem 4.5.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically compact;*
(2) *$\mathcal{R}$ is compact and directedly convergence-adherence complete.*

Moreover, from Th. 4.5, by Th. 3.6, it is clear that we also have

**Theorem 4.6.** *If $\mathcal{R}$ is a relator on $X$, with* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically compact;*
(2) *each relator on $X$ which is topologically equivalent to $\mathcal{R}$ is directedly convergence-adherence complete.*

**Remark 4.7.** Note that this theorem is only a partial extension of [20, Th. 2.2] of Sieber and Pervin, which is also only a partial extension of the famous Niemytzki–Tychonoff theorem [19].

## 5. Covering characterizations of topological compactness

**Theorem 5.1.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically compact;*
(2) *if $(A_\alpha)$ is a decreasing directed net of nonvoid subsets of $X$, then $\bigcap_\alpha A_\alpha^- \neq \emptyset$;*
(3) *if $(A_\alpha)$ is a increasing directed net of proper subsets of $X$, then $\bigcup_\alpha A_\alpha^\circ \neq X$.*

**Proof.** If $(x_\alpha)$ is a directed net in $X(\mathcal{R})$, then the family $(A_\alpha)$ of the sets $A_\alpha = \{x_\beta\}_{\beta \geq \alpha}$ is a decreasing directed net of nonvoid subsets of $X(\mathcal{R})$. Moreover, by the corresponding definitions, we have

$$\operatorname{adh}_{\mathcal{R}}(x_\alpha) = \bigcap_\alpha A_\alpha^- .$$

Hence, by Th. 4.3, it is clear that (2) implies (1).

On the other hand, if $(A_\alpha)$ is a decreasing directed net of nonvoid subsets of $X(\mathcal{R})$, then by choosing a point $x_\alpha$ in each $A_\alpha$ we can at once state that $(x_\alpha)$ is is a directed net in $X(\mathcal{R})$ such that

$$\operatorname{adh}_{\mathcal{R}}(x_\alpha) = \bigcap_\alpha \left(\{x_\beta\}_{\beta \geq \alpha}\right)^- \subset \bigcap_\alpha A_\alpha^- .$$

Hence, again by Th. 4.3, it is clear that (1) also implies (2).

Finally, we note that the equivalence of the assertions (2) and (3) is an immediate consequence of the De Morgan's laws and the simple fact that $X \setminus A^\circ = (X \setminus A)^-$ for all $A \subset X$. ◊

This intermediate theorem allows us to easily get the following more familiar

**Theorem 5.2.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically compact;*
(2) *each centred family $\mathcal{A}$ of subsets of $X(\mathcal{R})$ satisfies $\bigcap \mathcal{A}^- \neq \emptyset$;*
(3) *each interior cover $\mathcal{B}$ of $X(\mathcal{R})$ has a finite subcover.*

**Proof.** If $(A_\alpha)$ is a decreasing directed net of nonvoid subsets of $X(\mathcal{R})$, then the set $\{A_\alpha\}$ is a centred family of nonvoid subsets of $X(\mathcal{R})$. Hence, by Th. 5.1, it is clear that (2) implies (1).

On the other hand, if $\mathcal{A}$ is a centred family of subsets of $X(\mathcal{R})$, then $\mathcal{A}'$ is a nonvoid directed set of nonvoid subsets of $X(\mathcal{R})$ with respect to the reverse set inclusion such that $\mathcal{A} \subset \mathcal{A}'$. Thus, by considering the net $(A)_{A \in \mathcal{A}'}$, we can again apply Th. 5.1 to show that (1) also implies (2).

The equivalence of the assertions (2) and (3) is again an immediate consequence of the De Morgan's laws and the fact that a subset $X(\mathcal{R})$ is open if and only if its complement is closed. ◊

From Th. 5.2, by [22, Th. 2.3], it is clear that in particular we also have

**Theorem 5.3.** *If $\mathcal{R}$ is a topological relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically compact;*
(2) *each centred family $\mathcal{A}$ of closed subsets of $X(\mathcal{R})$ has a nonvoid intersection;*
(3) *each open cover $\mathcal{B}$ of $X(\mathcal{R})$ has a finite subcover.*

**Remark 5.4.** By the corresponding theorems, it is clear that the implications (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) do not require $\mathcal{R}$ to be topological.

On the other hand, the next simple example shows that the implication (3) $\Longrightarrow$ (1) does not, in general, hold if $\mathcal{R}$ is not topological.

**Example 5.5.** If $X$ is the set of all real numbers and $R \subset X \times X$ such that

$$R(x) = ]-\infty, x] \cup \{x+1\}$$

for all $x \in X$, then $\mathcal{R} = \{R\}$ is a noncompact relator on $X$ such that $\emptyset$ and $X$ are the only open subsets of $X(\mathcal{R})$.

## 6. Denseness conditions implying compactnesses

A simple reformulation of Def. 3.1 gives

**Theorem 6.1.** *If $\mathcal{R}$ is a relator on $X$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is compact;*

(2) *each of the spaces $X(\{R\})$, where $R \in \mathcal{R}$, has a finite inversely dense subset.*

**Proof.** Because of the corresponding definitions, we evidently have

$$\mathrm{cl}_{\{R\}^{-1}}(A) = R(A)$$

for all $A \subset X$, and hence the equivalence of (1) and (2) is immediate. $\Diamond$

Hence, it is clear that in particular we also have

**Corollary 6.2.** *A strongly symmetric relator $\mathcal{R}$ on $X$ is compact if and only if each of the spaces $X(\{R\})$, where $R \in \mathcal{R}$, has a finite dense subset.*

Moreover, as another easy consequence of Def. 3.1, we can also state

**Theorem 6.3.** *If a relator space $X(\mathcal{R})$ has a finite inversely dense subset $A$, then $X(\mathcal{R})$ is is proximally compact.*

**Proof.** Because of the corresponding definitions, now we evidently have

$$X = \mathrm{cl}_{\mathcal{R}^{-1}}(A) = \bigcap_{R \in \mathcal{R}} R(A).$$

This implies that $R(A) = X$ for all $R \in \mathcal{R}$. And therefore, we also have $S(A) = X$ for all $S \in \mathcal{R}^{\#}$. $\Diamond$

Hence, it is clear that in particular we also have

**Corollary 6.4.** *A topologically symmetric relator space $X(\mathcal{R})$ having a finite dense subset is proximally compact.*

The fact that this sufficient condition is very far from being necessary is apparent from the next simple

**Example 6.5.** If $X = [0,1]$ and for each positive integer $n$ we set

$$R_n = \{(x,y) \in X \times X : \ |x-y| < 1/n\},$$

then $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ is a strongly symmetric and topologically compact relator on $X$ such that the space $X(\mathcal{R})$ has no finite dense subset.

Moreover, as an important addition to Th. 4.3, we can also prove

**Theorem 6.6.** *If an uniformly filtered and topologically transitive relator space $X(\mathcal{R})$ has an inversely dense subset $A$ such that each directed net in $A$ is adherent in $X(\mathcal{R})$, then $X(\mathcal{R})$ is topologically compact.*

**Proof.** Note that if $A$ is in inversely dense subset of $X(\mathcal{R})$, then

$$X = \mathrm{cl}_{\mathcal{R}^{-1}}(A) = \bigcap_{R \in \mathcal{R}} R(A).$$

Therefore, if $(x_\alpha)_{\alpha \in \Gamma}$ is a net in $X$, then for each $\alpha \in \Gamma$ and $R \in \mathcal{R}$ there exists a point $y_{(\alpha,R)}$ in $A$ such that

$$x_\alpha \in R\left(y_{(\alpha,R)}\right).$$

Moreover, if $\Gamma$ is directed, then because of the uniform filteredness of $\mathcal{R}$ the family

$$\left(y_{(\alpha,R)} : \alpha \in \Gamma, R \in \mathcal{R}\right)$$

can be made into a directed net by setting

$$(\alpha, R) \leq (\alpha, R) \iff \alpha \leq \beta \mathrm{and} S \subset R.$$

Therefore, if each directed net in $A$ is adherent in $X(\mathcal{R})$, then there exists a point $x$ in $X$ such that

$$x \in \mathrm{adh}_{\mathcal{R}}\left(y_{(\alpha,R)}\right).$$

Hence, because of Th. 4.3, it remains only to show that now we also have $x \in \mathrm{adh}_{\mathcal{R}}(x_\alpha)$. For this, note that if $R_0 \in \mathcal{R}$, then because of the topological transitiveness of $\mathcal{R}$, there exist $U, V \in \mathcal{R}$ such that

$$V(U(x)) \subset R_0(x).$$

Moreover, if $\alpha_0 \in \Gamma$, then because of $x \in \mathrm{adh}_{\mathcal{R}}\left(y_{(\alpha,R)}\right)$ there exist $\alpha \in \Gamma$ and $R \in \mathcal{R}$, with $\alpha \geq \alpha_0$ and $R \subset V$, such that

$$y_{(\alpha,R)} \in U(x).$$

Hence, since

$$x_\alpha \in R\left(y_{(\alpha,R)}\right),$$

it is clear that

$$x_\alpha \in R\left(U(x)\right) \subset V\left(U(x)\right) \subset R_0(x). \quad \Diamond$$

**Remark 6.7.** Note that a relator space $X(\mathcal{R})$ can have an inversely dense subset if and only if it is inversely non-partial.

From Th. 6.6, it is clear that in particular we also have

**Corollary 6.8.** *An uniformly filtered, topologically transitive and topologically symmetric relator space $X(R)$ is topologically compact if it has a dense subset $A$ such that each directed net in $A$ is adherent in $X(\mathcal{R})$.*

**Remark 6.9.** Hence, by using Davis [4, Th. 4], one can easily get a slight improvement of Gál's theorem [9].

## 7. A few supplementary notes and comments

Because of the corresponding definitions of Murdeshwar–Naimpally [16, pp. 48–49] and Fletcher–Lindgren [7, pp. 12 and 51], a compact relator may be called precompact, but cannot be called totally bounded.

The use of the term "compact" instead "precompact" and the second part of Def. 3.1 are well motivated by our former treatments of the various continuities and connectednesses [21] and [13].

Th. 3.6 is closely related to [20, Lemma 2.1] of Sieber and Pervin. Note that according to [24, Th. 9.6] each reasonable generalized closure can be derived from a relator.

Th. 4.1 greatly extends and supplements not only the corresponding theorems of Kelley [11, p. 189] and Sieber and Pervin [20], but also those of Davis [5] and Konishi [12, p. 170].

The useful idea of deriving Th. 4.3 from Th. 4.1 has also been suggested by Konishi [12]. It reveals a remarkable advantage of relator spaces over quasi-uniform spaces.

Because of a striking proof of Frank [8], it is possible that Th. 4.1 can also be derived from Th. 4.3. This derivation however seems now to be quite artificial in the light of our present treatment.

Traditionally, a very particular case of Th. 4.3 is derived from that of Th. 5.3. And the corresponding particular cases of Theorems 5.1 and 5.2 remain usually annoticed.

Th. 5.2 came directly from the statements 41 A.9 and (5.3.1) of Čech [1, p. 783] and Császár [3, p. 193], who also applied some reverses of Kelley's treatment [11, p. 136].

The proof of Th. 6.6 is largely based upon that of Davis [5, Th. 5]. Despite that regularity is necessary in Gál's theorem [9], the conditions of Th. 6.6 and its corollary can still be weakened.

Finally, we remark that according to [27], [15] and [28] there are some further important refinements and modifications of a relator. Therefore, some further particular cases of the present definition of compactness should also be investigated.

Moreover, the corresponding $\Lambda$-compact relators, where $\Lambda$ is a suitable family of cardinal numbers, should also be investigated. Note that a relator $\mathcal{R}$ on $X$ may be called $\Lambda$-compact if for each $R \in \mathcal{R}$ there exists set $A \subset X$, with $\operatorname{card}(A) \in \Lambda$, such that $R(A) = X$.

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