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ON SOME CURVES IN VECTOR LATTICES

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Abstract: We deal with concave mappings F from an interval I on the real line into a boundedly complete vector lattice $(X, \preceq)$ such that for a given $0 \preceq \varepsilon \in X$ the following inequality

$$\lambda F(x) + (1-\lambda)F(y) \preceq F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y) + \varepsilon,$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. We look for affine functions $a : I \longrightarrow X$ that separate F and $F + \varepsilon$ in the sense that $F(x) \preceq a(x) \preceq F(x) + \varepsilon$ for all $x \in I$ ("sandwich" type theorems). Some of the results are obtained under an additional assumption that the vector lattice in question is a Banach lattice. Extension theorems for vector-valued concave mappings as well as links with the Hyers-Ulam stability problem for affine transformations are also presented.

1. Introduction

The stability question for convex mappings was first investigated by D. H. Hyers and S. Ulam [6] in 1952. They have proved that given a nonempty open and convex set $D \subset \mathbb{R}^n$, an $\varepsilon \geq 0$ and a function $f : D \longrightarrow \mathbb{R}$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \varepsilon,$$

for all $x, y \in D$ and all $\lambda \in [0,1]$ there exists a convex function $g : D \longrightarrow \longrightarrow \mathbb{R}$ such that

$$|f(x) - g(x)| \leq k_n \cdot \varepsilon, \quad x \in D,$$

where k_n is a constant depending exclusively upon the dimension of the space $\mathbb{R}^n$ considered.

In 1984 P. W. Cholewa [3] gave a simpler proof of this result and considerably improved the constants k_n. An interesting generalization of that stability problem has recently been considered by K. Baron, J. Matkowski and K. Nikodem in [1]. Some futher authors were involved but generally their endeavours were concerned with either Jensen approximate convexity or with infinite dimensional domains. However, all these stability considerations were concerned with scalar functions exclusively. In the present paper, we give some results on concave vector-valued solutions to inequality

$$(1) \qquad F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda) F(y) + \varepsilon,$$

where x, y run over an interval $I \subset \mathbb{R}, \lambda \in [0,1]$ and ε is a given element of the positive cone in a boundedly complete vector lattice. In the sequel, solutions of inequality (1) will be referred to as *ε-convex* mappings. Concave ε-convex mappings are then characterized by

$$0 \preceq F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda) F(y) \preceq \varepsilon,$$

assumed for all $x, y \in I$ and all $\lambda \in [0,1]$. Plainly, such a requirement is considerably stronger than ε-convexity itself. However, our aim will be now to approximate solutions of the latter inequality by *affine* ones. More precisely, we are going to find mappings a satisfying the functional equation

$$a(\lambda x + (1-\lambda)y) = \lambda a(x) + (1-\lambda) a(y)$$

for every $x, y \in I$, $\lambda \in [0,1]$ and such that

$$0 \preceq a(x) - F(x) \preceq \varepsilon,$$

for all $x \in I$. Consequently, we deal with Hyers–Ulam stability problem for affine mappings in the spirit proposed recently by K. Nikodem and Sz. Wąsowicz [9] in the class of real functions of one real variable.

It turns out that the case where the interval I in question is open and bounded the stability problem just described happens to be the most delicate one. This forced us to look for possible extensions of a given vector-valued concave mapping on I onto at least half-closed interval. Such type results are presented in the final part of Section 3.

2. Preliminaries

We call the collection of all nonnegative elements in a given vector lattice the *positive cone*. A vector lattice X is called *boundedly complete* if every nonempty order majorized subset of X has a supremum. A real Banach space $(X, \|\cdot\|)$ is termed a *Banach lattice* whenever X is a vector lattice with the partial order $\preceq$ such that

$$|u| \preceq |v| \quad \text{implies} \quad \|u\| \leq \|v\|, \quad u, v \in X,$$

where $|u| := \sup\{u, -u\}$, $u \in X$. Positive cones in Banach lattices are closed in the norm topology.

We write X_+^* for the *dual cone* in a Banach lattice $(X, \|\cdot\| \preceq)$, i.e. the totality of positive continuous linear real functionals on X. Each member of the dual space X^* can be decomposed into a difference of two elements of X_+^*; in other words: $X^* = X_+^* - X_+^*$. An element x in a Banach lattice $(X, \|\cdot\|, \preceq)$ is nonnegative if and only if $p(x) \geq 0$ for all $p \in X_+^*$.

The positive cone in a Banach lattice $(X, \|\cdot\|, \preceq)$ is said to be *regular* in the sense of M. A. Krasnoselskij (see [7]) provided that each order increasing sequence of elements from X, order bounded above is norm convergent. For example, given a real Banach space $(X, \|\cdot\|)$, a vector $x \in X \setminus \{0\}$ and a number $\varrho \in (0, \|x\|)$ by setting

$$C := \bigcup \{\lambda \operatorname{cl} B(x, \varrho) : \lambda \geq 0\},$$

where $B(x, \varrho)$ stands for the open ball centered at x and having radius ϱ, we obtain a regular cone in a Banach lattice $(X, \|\cdot\|, \preceq)$ with

$$u \preceq v \quad \text{if and only if} \quad v - u \in C, \quad u, v \in X$$

Clearly, in that case $\operatorname{int} C$ is nonvoid. Moreover, a Banach lattice obtained in that way is boundedly complete because each Banach lattice with regular positive cone is automatically boundedly complete (cf. H.-U. Schwarz [11, Chapter II, Propositions 3.2 and 3.3]).

We refer the reader to the book just quoted as well as to the classical monograph of G. Birkhoff [2] for further notions and facts on lattice theory.

3. Main results

We begin with a result that provides a description of the analytic form of a composition of a concave and ε-convex mapping with a member of the dual cone.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval (bounded or not) and let $(X, \|\cdot\|, \preceq)$ be a Banach lattice. Given an $X \ni \varepsilon \succeq 0$ assume that $F : I \longrightarrow X$ is a concave mapping satisfying inequality* (1) *for all $x, y \in I$ and all $\lambda \in [0,1]$. Then for every member p of X_+^* there exists an affine function $a_p : I \longrightarrow \mathbb{R}$ and a convex function $\gamma_p : I \longrightarrow [0,1]$ such that*

$$p \circ F = a_p - \gamma_p \cdot p(\varepsilon)$$

Proof. For every $p \in X_+^*$ one has

$$p\left(F(\lambda x + (1-\lambda)y)\right) \le \lambda p\left(F(x)\right) + (1-\lambda)p\left(F(y)\right) + p(\varepsilon),$$

$x, y \in I$, $\lambda \in [0,1]$. This means that the superposition $p \circ F : I \to \to \mathbb{R}$ satisfies the assumptions of the celebrated Hyers–Ulam stability theorem [6] (see also P. Cholewa [3] and K. Baron, J. Matkowski and K. Nikodem [1]). Thus there exists a convex function $f_p : I \to \mathbb{R}$ such that

$$p\left(F(x)\right) \le f_p(x) \le p\left(F(x)\right) + p(\varepsilon), \quad x, y \in I. \tag{2}$$

In particular, a concave function $p \circ F$ is majorized by f_p which is convex. On account of G. Rodé's result [10, Beispiel (b)] there exists an affine function $a_p : I \to \mathbb{R}$ such that

$$p\left(F(x)\right) \le a_p(x) \le f_p(x), \quad x \in I$$

By means of (2) we have also

$$p\left(F(x)\right) \le a_p(x) \le p\left(F(x)\right) + p(\varepsilon), \quad x, y \in I,$$

which implies the existence of a function $\gamma_p : I \to [0,1]$ such that

$$a_p(x) = (1 - \gamma_p(x))\, p\left(F(x)\right) + \gamma_p(x)\left[p\left(F(x)\right) + p(\varepsilon)\right], x \in I,$$

whence

$$p(\varepsilon) \cdot \gamma_p(x) = a_p(x) - p\left(F(x)\right), \quad x \in I.$$

Therefore, γ_p is convex and

$$p\left(F(x)\right) = a_p(x) - \gamma_p(x) \cdot p(\varepsilon), \quad x \in I. \quad \Diamond$$

From Th. 1 we obtain the following two corollaries:

Corollary 1. *Under the assumptions of* Th. 1 *any concave solution of inequality* (1) *on $\mathbb{R}$ has to be an affine function, i.e. if $F : \mathbb{R} \longrightarrow X$ is concave and ε-convex, then*

$$F\left(\lambda x + (1-\lambda)y\right) = \lambda F(x) + (1-\lambda)F(y),$$

for all $x, y \in \mathbb{R}$ and all $\lambda \in [0,1]$.

Proof. It suffices to observe, that each convex function $\gamma : \mathbb{R} \to [0,1]$ is necessarily constant and the dual cone X_+^* is total. $\Diamond$

A scalar function is called delta-convex provided that it can be represented as a difference of two convex functions. A function F with values in a Banach space X is called weakly delta-convex iff $x^* \circ F$ is delta-convex for all $x^* \in X^*$.

Corollary 2. *Under the assumptions of* Th. 1 *any concave solution to* (1) *is weakly delta-convex.*

Proof. Take $x^* \in X^*$ and decompose it into a difference $p_1 - p_2$ of two members of X^*_+. An appeal to Th. 1 gives the existence of convex functions $\gamma_{p_i} : I \to [0,1], i \in \{1,2\}$, such that

$$x^*(F(x)) = (a_{p_1}(x) + \gamma_{p_2}(x) \cdot p(\varepsilon)) - (a_{p_2}(x) + \gamma_{p_1}(x) \cdot p(\varepsilon)), x \in I.$$

Since the functions $a_{p_1} + \gamma_{p_2} \cdot p(\varepsilon)$ and $a_{p_2} + \gamma_{p_1} \cdot p(\varepsilon)$ are convex, the proof has been completed. ◊

Remark 1. *Let* $(X, \|\cdot\|, \preceq)$ *be a Banach lattice with a strong unit (see e.g. G. Birkhoff [2]). Any concave mapping from an interval* $I \subset \mathbb{R}$ *into* X *is automatically continuous in* $\operatorname{int} I$.

Proof. Let $F : I \to X$ be a concave mapping. Obviously, for every $p \in X^*_+$, the function $p \circ F : I \to \mathbb{R}$ is concave and hence continuous in $\operatorname{int} I$ (see, for instance, M. Kuczma [8]). It suffices to apply Th. 2 from the second author's paper [4]. ◊

In the case where the interval I is a closed subset of $\mathbb{R}$ we get the desired separation effect. The next theorem will show it up.

Theorem 2. *Let* $(X, \preceq)$ *be a vector lattice and let* $\alpha, \beta \in \mathbb{R}, \alpha < \beta$. *If* $F : [\alpha, \beta] \longrightarrow X$ *is a concave function such that inequality*

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y) + \varepsilon,$$

is satisfied for some $\varepsilon \succeq 0$ *and for all* $x, y \in I, \lambda \in [0,1]$, *then an affine function*

$\phi : [\alpha, \beta] \longrightarrow X$ *given by the formula*

$$\phi(x) := \frac{F(\beta) - F(\alpha)}{\beta - \alpha}(x - \alpha) + F(\alpha) + \varepsilon, \quad x \in [\alpha, \beta],$$

separates F *and* $F + \varepsilon$, *i.e.*

$$F(x) \preceq \phi(x) \preceq F(x) + \varepsilon, \quad x \in [\alpha, \beta].$$

Proof. For every $x \in [\alpha, \beta]$ there exists a $\lambda \in [0,1]$ such that $x = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}\phi(x) =& \phi(\lambda\alpha + (1-\lambda)\beta) = \frac{F(\beta) - F(\alpha)}{\beta - \alpha}(1-\lambda)(\beta - \alpha) + F(\alpha) + \varepsilon \\ =& \lambda F(\alpha) + (1-\lambda)F(\beta) + \varepsilon \preceq F(x) + \varepsilon.\end{aligned}$$

On the other hand, from the equality $1 - \lambda = \frac{x-\alpha}{\beta-\alpha}$ and from inequality (1) we infer that

$$F(x) - F(\alpha) \preceq (1-\lambda)\,(F(\beta) - F(\alpha)) + \varepsilon = \tag{3}$$
$$= \frac{x-\alpha}{\beta-\alpha}\,(F(\beta) - F(\alpha)) + \varepsilon,$$

which is equivalent to

$$F(x) \preceq \frac{F(\beta) - F(\alpha)}{\beta - \alpha}(x - \alpha) + F(\alpha) + \varepsilon = \phi(x). \quad \Diamond$$

Inequality (3) in the latter proof leads easily to the following

Remark 2. *Assume that $I \subset \mathbb{R}$ is an interval, $F : I \to X$ is ε-convex and $\alpha, \gamma \in I, \alpha < \gamma$, are arbitrary. Then for every $t \in [\alpha, \gamma]$ the following inequality is fulfilled:*

$$\frac{F(t) - F(\alpha)}{t - \alpha} \preceq \frac{F(\gamma) - F(\alpha)}{\gamma - \alpha} + \frac{\varepsilon}{t - \alpha} \quad \text{for} \quad t \in (\alpha, \gamma].$$

Now, we shall consider the situation where $I = [\alpha, \beta), \alpha < \beta \leq \leq \infty$, or $I = (\beta, \alpha], -\infty \leq \beta < \alpha$, for some $\alpha \in \mathbb{R}$. We start with the following two lemmas.

Lemma 1. *Let $(X, \preceq)$ be a boundedly complete vector lattice and let $\alpha \in \mathbb{R}$. Let further I stand for a half-open interval $[\alpha, \beta)$ or $(\beta, \alpha]$ (infinite β admissible). If $F : I \to X$ is a concave solution to inequality* (1) *then the elements*

$$\inf\left\{\frac{F(x) - F(\alpha)}{x - \alpha} : x \in (\alpha, \beta)\right\} \quad \text{if} \quad I = [\alpha, \beta)$$

and

$$\sup\left\{\frac{F(x) - F(\alpha)}{x - \alpha} : x \in (\beta, \alpha)\right\} \quad \text{if} \quad I = (\beta, \alpha]$$

are well-defined.

Proof. Assume that $I = [\alpha, \beta)$ and fix an $x_0 \in (\alpha, \beta)$. Then, for any $x \in [x_0, \beta)$ we have $x_0 \in (\alpha, x]$ and from Remark 2 applied for $\gamma = x$ and $t = x_0$ we get

$$\frac{F(x) - F(\alpha)}{x - \alpha} \succeq \frac{F(x_0) - F(\alpha)}{x_0 - \alpha} - \frac{\varepsilon}{x_0 - \alpha}.$$

Since the space is assumed to be boundedly complete there exists

$$\inf\left\{\frac{F(x) - F(\alpha)}{x - \alpha} : x \in [x_0, \beta)\right\}. \tag{4}$$

On the other hand, if $x \in (\alpha, x_0)$, then from the concavity of F we obtain the inequality

$$\frac{F(x) - F(\alpha)}{x - \alpha} \succeq \frac{F(x_0) - F(\alpha)}{x_0 - \alpha},$$

which guarantees that

$$\inf\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in(\alpha,x_0]\right\} \tag{5}$$

does exist. From (4) and (5) it follows that

$$a := \inf\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in(\alpha,\beta)\right\}.$$

exists in X. In the case where $I=(\beta,\alpha]$ the proof is quite analogous. ◇

Lemma 2. *Let the assumptions of* Lemma 1 *be fulfilled. Then for every $x_0 \in I$ one has*

$$\inf\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in(\alpha,\beta)\right\} = \inf\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in[x_0,\beta)\right\} \quad \text{if } I=[\alpha,\beta);$$

$$\sup\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in(\beta,\alpha)\right\} = \sup\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in(\beta,x_0]\right\} \quad \text{if } I=(\beta,\alpha].$$

Proof. We shall examine the case where $I=[\alpha,\beta)$ in detail. In the other case the proof is the same. Fix an $x\in(\alpha,\beta)$ and put

$$a := \inf\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in(\alpha,\beta)\right\}$$

and

$$c := \inf\left\{\frac{F(x)-F(\alpha)}{x-\alpha} : x\in[x_0,\beta)\right\}.$$

We have $a \preceq c$. Clearly,

$$c \preceq \frac{F(x)-F(\alpha)}{x-\alpha} \quad \text{for all} \quad x\in[x_0,\beta),$$

whereas for $x\in(\alpha,x_0]$ the concavity of F gives

$$c \preceq \frac{F(x_0)-F(\alpha)}{x_0-\alpha} \preceq \frac{F(x)-F(\alpha)}{x-\alpha}.$$

Thus $c \preceq a$, which completes the proof. ◇

With the aid of these two lemmas we shall prove the following

Theorem 3. *Given a real number α, a boundedly complete vector lattice $(X,\preceq)$ and a vector $\varepsilon \succeq 0$ from X, let I stand for the interval $[\alpha,\beta)$ or $(\beta,\alpha]$ (infinite β admissible). If $F : I \longrightarrow X$ is a concave solution to* (1)*, then there exists a vector $a\in X$ such that*

$$F(x) \preceq a(x-\alpha)+F(\alpha)+\varepsilon \preceq F(x)+\varepsilon,$$

for all $x\in I$.

Proof. Assume that $I = [\alpha, \beta)$. Lemma 1 implies that

$$a := \inf \left\{ \frac{F(x) - F(\alpha)}{x - \alpha} : x \in (\alpha, \beta) \right\}$$

does exist in X. Hence, for every $x \in [\alpha, \beta)$ we get

$$F(x) \succeq a(x - \alpha) + F(\alpha). \tag{6}$$

Take an $x \in (\alpha, \beta)$ and a $y \in [x, \beta)$. From Remark 2 applied for $\gamma = y$ and $t = x$ we deduce that

$$\frac{F(x) - F(\alpha)}{x - \alpha} \preceq \frac{F(y) - F(\alpha)}{y - \alpha} + \frac{\varepsilon}{x - \alpha};$$

therefore,

$$\frac{F(x) - F(\alpha)}{x - \alpha} \preceq \inf \left\{ \frac{F(y) - F(\alpha)}{y - \alpha} : y \in [x, \beta) \right\} + \frac{\varepsilon}{x - \alpha}.$$

Now, Lemma 2 implies that

$$\frac{F(x) - F(\alpha)}{x - \alpha} \preceq a + \frac{\varepsilon}{x - \alpha}.$$

In other words

$$F(x) \preceq a(x - \alpha) + F(\alpha) + \varepsilon$$

for every $x \in [\alpha, \beta)$, which jointly with (6) finishes the proof, because the case where $I = (\beta, \alpha]$ can be handled in much the same way.

Surprisingly, the most sophisticated is the case where the interval considered is a bounded and open subset of real line. To proceed we begin with the following

Lemma 3. *Let $I = (\alpha, \beta), \alpha, \beta, \in \mathbb{R}, \alpha < \beta$, and let $(X, \preceq)$ be a boundedly complete vector lattice. If $F : (\alpha, \beta) \longrightarrow X$ is a concave solution of inequality* (1)*, then for every $x_0 \in (\alpha, \beta)$ the elements*

$$a^-(x_0) := \sup \left\{ \frac{F(x) - F(x_0)}{x - x_0} : x \in (\alpha, x_0) \right\}$$

$$b^-(x_0) := \inf \left\{ \frac{F(x) - F(x_0)}{x - x_0} : x \in (\alpha, x_0) \right\}$$

$$a^+(x_0) := \sup \left\{ \frac{F(x) - F(x_0)}{x - x_0} : x \in (x_0, \beta) \right\}$$

$$b^+(x_0) := \inf \left\{ \frac{F(x) - F(x_0)}{x - x_0} : x \in (x_0, \beta) \right\}$$

are correctly defined.

Proof. Fix arbitrary x_0, y_0 and η such that $\alpha < x_0 < y_0 < \beta$ and $\eta \in (0, x_0 - y_0)$ and take an $x \in (\alpha, x_0)$. There exists a $y \in (\alpha, x)$ such that $x_0 - y \geq \eta$.

We get $\alpha < y < x < x_0 < y_0 < \beta$. From the concavity of F we obtain the following inequalities:

$$\frac{F(y_0)-F(x_0)}{y_0-x_0} \preceq \frac{F(x)-F(x_0)}{x-x_0} \preceq \frac{F(y)-F(x_0)}{y-x_0}. \tag{7}$$

We shall show that

$$\frac{F(y)-F(x_0)}{y-x_0} \preceq \frac{F(y_0)-F(x_0)}{y_0-x_0} + \frac{\beta-\alpha}{\eta(y_0-x_0)}\varepsilon. \tag{8}$$

Indeed, since

$$x_0 = \frac{y_0-x_0}{y_0-y}y + \frac{x_0-y}{y_0-y}y_0,$$

inequality (1) applied for $\lambda := \frac{y_0-x_0}{y_0-y} \in (0,1)$ leads to

$$F(x_0) \preceq \frac{y_0-x_0}{y_0-y}F(y) + \frac{x_0-y}{y_0-y}F(y_0) + \varepsilon.$$

Multiplying both sides of the latter inequality by $\frac{y_0-y}{x_0-y}\cdot\frac{1}{y_0-x_0}$ and taking into account that

$$\frac{y_0-y}{x_0-y}\frac{1}{y_0-x_0} \leq \frac{\beta-\alpha}{\eta(y_0-x_0)}$$

we arrive at (8). Now, inequalities (7) and (8) imply that

$$\frac{F(y_0)-F(x_0)}{y_0-x_0} \preceq \frac{F(x)-F(x_0)}{x-x_0} \preceq \frac{F(y_0)-F(x_0)}{y_0-x_0} + \frac{\beta-\alpha}{\eta(y_0-x_0)}\varepsilon.$$

This clearly forces the existence of the elements $a^-(x_0)$ and $b^-(x_0)$. In the remaining cases the proof is literally the same. $\lozenge$

Corollary 4. *If $(X, \|\cdot\|, \preceq)$ is a boundedly complete Banach lattice, $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, and $F : (\alpha, \beta) \to X$ is a concave solution of inequality (1), then F is continuous.*

Proof. Let $x_0 \in (\alpha, \beta)$ be fixed. From Lemma 3 we infer that

$$b^-(x_0) \preceq \frac{F(x)-F(x_0)}{x-x_0} \preceq a^-(x_0) \quad \text{for all} \quad x \in (\alpha, x_0)$$

and

$$b^+(x_0) \preceq \frac{F(x)-F(x_0)}{x-x_0} \preceq a^+(x_0) \quad \text{for all} \quad x \in (x_0, \beta).$$

Put

$$c(x_0) := \sup\{a^-(x_0), -b^-(x_0)\}$$

and

$$d(x_0) := \sup\{a^+(x_0), -b^+(x_0)\}.$$

Since we have

$$\left|\frac{F(x)-F(x_0)}{x-x_0}\right| \preceq c(x_0) = |c(x_0)| \quad \text{for all} \quad x \in (\alpha, x_0)$$

and

$$\left|\frac{F(x)-F(x_0)}{x-x_0}\right| \preceq d(x_0) = |d(x_0)| \quad \text{for all} \quad x \in (x_0, \beta),$$

the estimations

$$\|F(x)-F(x_0)\| \leq |x-x_0|\|c(x_0)\|, \quad x \in (\alpha, x_0),$$

and

$$\|F(x)-F(x_0)\| \leq |x-x_0|\|d(x_0)\|, \quad x \in (x_0, \beta),$$

follow. Consequently, we get

$$\lim_{x\to x_0} F(x) = F(x_0)$$

which is our claim. $\Diamond$

The following theorem concerns the situation where the domain of a vector-valued map considered is an open and bounded subinterval of the real line.

Theorem 4. *Given real numbers $\alpha, \beta, \alpha < \beta$, a boundedly complete vector lattice $(X, \preceq)$ and a vector $\varepsilon \succeq 0$ from X, if $F : (\alpha, \beta) \longrightarrow X$ is a concave solution of inequality (1), then for every $x_0 \in (\alpha, \beta)$ there exists a hat-function $\varphi_{x_0} : (\alpha, \beta) \longrightarrow \mathbb{R}$ with the vertex at $(x_0, F(x_0)+\varepsilon)$ such that*

$$F(x) \preceq \varphi_{x_0}(x) \preceq F(x)+\varepsilon, \quad x \in (\alpha, \beta).$$

The function φ_{x_0} is of the form

$$\varphi_{x_0}(x) = \begin{cases} a^-(x_0)(x-x_0)+F(x_0)+\varepsilon, & \text{if} \quad x \in (\alpha, x_0], \\ b^+(x_0)(x-x_0)+F(x_0)+\varepsilon, & \text{if} \quad x \in (x_0, \beta). \end{cases}$$

Proof. Fix arbitrarily an $x_0 \in (\alpha, \beta)$ and observe, that the concavity of F forces the function

$$\varphi(x) := \frac{F(x)-F(x_0)}{x-x_0}, \quad x \in (\alpha, \beta),$$

to be decreasing, i.e. if $u \leq v$ then $\varphi(u) \succeq \varphi(v)$. Hence

$$\sup\{\varphi(t) : t \in (\alpha, u]\} = \sup\{\varphi(t) : t \in (\alpha, v]\}.$$

Take $x \in (\alpha, x_0]$ and $u \in (\alpha, x)$. There exists a $\lambda \in (0,1)$ such that $x = \lambda u + (1-\lambda)x_0$. From inequality (1) applied for $\lambda = \frac{x-x_0}{u-x_0}$ we get

$$F(x) \preceq \lambda F(u) + (1-\lambda)F(x_0)+\varepsilon = \frac{F(u)-F(x_0)}{u-x_0}(x-x_0)+F(x_0)+\varepsilon.$$

Hence

$$(9) \quad \begin{cases} F(x) \preceq \sup\left\{\dfrac{F(u)-F(x_0)}{u-x_0}(x-x_0) : u \in (\alpha, x)\right\} + F(x_0)+\varepsilon \\ \qquad = \sup\left\{\dfrac{F(u)-F(x_0)}{u-x_0}(x-x_0) : u \in (\alpha, x_0]\right\} + F(x_0)+\varepsilon \\ \qquad = a^-(x_0)(x-x_0)+F(x_0)+\varepsilon. \end{cases}$$

Since

$$a^-(x_0) \succeq \frac{F(x) - F(x_0)}{x - x_0}$$

we have

$$F(x) \succeq a^-(x_0)(x - x_0) + F(x_0) \quad \text{for} \quad x \in (\alpha, x_0],$$

which jointly with (9) implies that

$$F(x) \preceq a^-(x_0)(x - x_0) + F(x_0) + \varepsilon \preceq F(x) + \varepsilon \quad \text{for} \quad x \in (\alpha, x_0].$$

The monotonicity of φ gives

$$\inf\{\varphi(t) : t \in [u, \beta)\} = \inf\{\varphi(t) : t \in [v, \beta)\}$$

provided that $u \leq v$. Let $x \in [x_0, \beta)$ and $u \in (x, \beta)$. Along the same lines we show that

$$F(x) \preceq \frac{F(u) - F(x_0)}{u - x_0}(x - x_0) + F(x_0) + \varepsilon,$$

whence by Lemma 3 we arrive at:

(10)

$$\begin{cases} F(x) \preceq \inf\left\{\dfrac{F(u) - F(x_0)}{u - x_0}(x - x_0) : u \in (x, \beta)\right\} + F(x_0) + \varepsilon \\ \qquad = \inf\left\{\dfrac{F(u) - F(x_0)}{u - x_0}(x - x_0) : u \in [x_0, \beta)\right\} + F(x_0) + \varepsilon \\ \qquad = b^+(x_0)(x - x_0) + F(x_0) + \varepsilon. \end{cases}$$

Now, in view of the obvious inequality

$$\frac{F(x) - F(x_0)}{x - x_0} \succeq b^+(x_0),$$

we infer that

$$F(x) \succeq b^+(x_0)(x - x_0) + F(x_0) \quad \text{for} \quad x \in [x_0, \beta).$$

From this inequality by means of (10) we obtain

$$F(x) + \varepsilon \succeq b^+(x_0)(x - x_0) + F(x_0) + \varepsilon \succeq F(x) \quad \text{for} \quad x \in [x_0, \beta).$$

and the proof has been completed. ◊

We terminate this paper with stating some sufficient conditions for the existence of a continuous extension of a concave function on an open interval to a concave function on its closure. If that is the case then, in particular, we may apply separation theorems obtained earlier and to have an affine separating function on an open interval instead of hat-functions occurring in Th. 4. We also believe that such type extension results may present an interest of its own.

In what follows, by the graph of a function $F : (\alpha, \beta) \longrightarrow X$ we mean the curve

$$\operatorname{gr} F := \{(x, F(x)) \in \mathbb{R} \times X : x \in (\alpha, \beta)\}$$

whereas the symbol $|\operatorname{gr} F|$ stands for the length of that curve (if it is rectifiable).

Theorem 5. *Given real numbers $\alpha, \beta, \alpha < \beta$, a Banach lattice $(X, \|\cdot\|, \preceq)$ and a concave map $F : (\alpha, \beta) \longrightarrow X$, if either*

(a) $\operatorname{gr} F$ *is rectifiable*

or

(b) $\operatorname{gr} F$ *is precompact in* $\mathbb{R} \times X$

or

(c) F *is order bounded below and the positive cone in* X *is regular, then there exists a continuous extension of* F *to a concave mapping on* $[\alpha, \beta]$.

Proof. (a). We shall show that there exists a continuous extension of F onto the interval $(\alpha, \beta]$ (the proof of the existence of a continuous extension of F onto the interval $[\alpha, \beta)$ is entirely analogous). First, we shall show that for every $\eta > 0$ there exists a $\delta \in (0, \beta)$ such that

$$\|F(s) - F(t)\| < \eta \quad \text{for every} \quad s, t \in (\beta - \delta, \beta).$$

Suppose the contrary: there exists an $\eta_0 > 0$ such that for every $\delta \in (0, \beta - \alpha)$ one may find $s, t \in (\beta - \delta, \beta)$ such that inequality

$$\|F(s) - F(t)\| \geq \eta_0$$

holds true. Fix arbitrarily an $M > 0$ and take a number $N \in \mathbb{N}$ such that $N\eta_0 > M$. Let $\alpha_0 := \frac{\beta+\alpha}{2}$. There exist s_1, t_1 such that $\alpha_0 < s_1 < t_1 < \beta$ and $\|F(s_1) - F(t_1)\| \geq \eta_0$. Proceeding similarly, we may find elements $s_2 < t_2$ in the interval (t_1, β) such that $\|F(s_2) - F(t_2)\| \geq \eta_0$. By induction, we derive the existence of sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $\alpha_0 < s_n < t_n < s_{n+1} < t_{n+1} < \beta$ and $\|F(s_n) - F(t_n)\| \geq \eta_0$ for all $n \in \mathbb{N}$. Hence, for every $\mathbb{N} \ni m \geq N$ one has

$$\begin{aligned}
|\operatorname{gr}(F|_{[\alpha_0, t_m]})| &\geq \sum_{i=1}^{m} \|(t_i, F(t_i)) - (s_i, F(s_i))\| = \\
&= \sum_{i=1}^{m} \sqrt{(t_i - s_i)^2 + \|F(t_i) - F(s_i)\|^2} \geq \\
&\geq \sum_{i=1}^{N} \|F(t_i) - F(s_i)\| \geq N\eta_0 > M.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\infty > |\operatorname{gr}(F|_{[\alpha_0, \beta)})| &= \sup\left\{|\operatorname{gr}(F|_{[\alpha_0, \beta_0]})| : \beta_0 \in (\alpha_0, \beta)\right\} \\
&\geq \sup\left\{|\operatorname{gr}(F|_{[\alpha_0, t_m]})| : m \geq N\right\} > M.
\end{aligned}$$

Since $M > 0$ was quite arbitrary this contradicts assumption (a). So, in view of the metric completeness of the Banach space considered, we conclude that F has a limit at the right endpoint of its domain and, consequently, that F admits a continuous extension onto $(\alpha,\beta]$. Obviously, that extension yields a concave mapping.

(b). We give the proof only for the extension onto the right end-point of the interval (α,β). Since there exists an $M>0$ such that

$$\operatorname{cl}\operatorname{gr}F\subset[\alpha,\beta]\times\operatorname{cl}B(0,M),$$

we infer that

$$F(t)\in\operatorname{cl}B(0,M)\quad\text{for all}\quad t\in(\alpha,\beta).$$

We shall first show that

(11) $$\operatorname{cl}\operatorname{gr}F\cap(\{\beta\}\times\operatorname{cl}B(0,M))\neq\emptyset.$$

Indeed, otherwise, since $\operatorname{cl}\operatorname{gr}F$ is compact, $\operatorname{cl}B(0,M)$ is closed and, obviously, both sets are nonempty, the distance

$$\eta:=\operatorname{dist}(\operatorname{cl}\operatorname{gr}F,\{\beta\}\times\operatorname{cl}B(0,M))$$

would be positive. This contradicts (11) because, taking a $t\in(\alpha,\beta)$ such that $t+\eta>\beta$, one has $(t,F(t))\in\operatorname{cl}\operatorname{gr}F$ and $(\beta,F(t))\in\{\beta\}\times$ $\times\operatorname{cl}B(0,M)$, yielding

$$\|(t,F(t))-(\beta,F(t))\|=|t-\beta|<\eta,$$

which violates the definition of η.

Now, we are going to prove that the intersection spoken of in (11) is a singleton. Suppose the contrary: there exist $y\neq z$ such that

$$(\beta,y),(\beta,z)\in\operatorname{cl}\operatorname{gr}F\quad\text{and}\quad(\beta,y),(\beta,z)\in\operatorname{cl}B(0,M).$$

Since the dual cone $X_+{}^*$ distinguishes points of X there exists a member p of $X_+{}^*$ such that $p(y)\neq p(z)$. Without loss of generality we may assume that $p(y)>p(z)$. Clearly, there exist sequences $(\beta_n)_{n\in\mathbb{N}}$, $(\overline{\beta}_n)_{n\in\mathbb{N}}$, $\beta_n<\beta$, $\overline{\beta}_n<\beta$ such that

$$\lim_{n\to\infty}(\beta_n,F(\beta_n))=(\beta,y),\qquad\lim_{n\to\infty}(\overline{\beta}_n,F(\overline{\beta}_n))=(\beta,z)$$

and

$$\lim_{n\to\infty}p(F(\beta_n))=p(y),\qquad\lim_{n\to\infty}p(F(\overline{\beta}_n))=p(z).$$

Let $\eta>0$ be such that

(12) $$(p(y)-\eta,p(y)+\eta)\cap(p(z)-\eta,p(z)+\eta)=\emptyset.$$

Choose elements $\beta_{n_1},\beta_{n_3}\in\{\beta_n:n\in\mathbb{N}\}$ and $\overline{\beta}_{n_2}\in\{\overline{\beta}_n:n\in\mathbb{N}\}$ such that $\beta_{n_1}<\overline{\beta}_{n_2}<\beta_{n_3}$ and

$$p\left(F(\beta_{n_1})\right)\in(p(y)-\eta,p(y)+\eta),$$

$$p\left(F(\overline{\beta}_{n_2})\right)\in(p(z)-\eta,p(z)+\eta),$$

$$p\left(F(\beta_{n_3})\right)\in(p(y)-\eta,p(y)+\eta).$$

From the concavity of $p\circ F$ we deduce that for

$$\lambda:=\frac{\overline{\beta}_{n_2}-\beta_{n_3}}{\beta_{n_1}-\beta_{n_3}}$$

we have

$$(p(z)-\eta, p(z)+\eta) \ni p\left(F(\overline{\beta}_{n_2})\right) = p\left(F(\lambda\beta_{n_1}+(1-\lambda)\beta_{n_3})\right) \geq$$
$$\geq \lambda p\left(F(\beta_{n_1})\right)+(1-\lambda)p\left(F(\beta_{n_3})\right) \in (p(y)-\eta, p(y)+\eta),$$

which contradicts (12). Thus, there exits an element $g \in X$ such that

$$\operatorname{cl}\operatorname{gr} F \cap (\{\beta\} \times \operatorname{cl} B(0,M)) = \{(\beta, g)\}.$$

It remains to put

$$\tilde{F}(x) := \begin{cases} F(x) & \text{for} \quad x \in (\alpha,\beta) \\ g & \text{for} \quad x = \beta \end{cases}$$

to get a continuous extension $\tilde{F}$ of F onto the interval $(\alpha, \beta]$. Plainly, $\tilde{F}$ is concave.

(c). Again, we shall confine ourselves to the right endpoint of the domain. Assume that there exists an m such that $F(x) \succeq m$ for x from a neighbourhood $[\gamma, \beta) \subset (\alpha, \beta)$ of β . Then the difference $F - m$ is a concave solution of inequality (1) and $F(x) - m \succeq 0$ for all $x \in [\gamma, \beta)$. Therefore, without loss of generality, we may assume that F itself is nonnegative in $[\gamma, \beta)$.

Now, fix arbitrarily points $s, t \in (\gamma, \beta), s < t$. By means of the concavity of F we obtain

$$\frac{F(t)-F(s)}{t-s} \preceq \frac{F(s)-F(\gamma)}{s-\gamma}.$$

Setting $\lambda := \frac{s-\gamma}{t-s}$ we get further

$$\lambda F(t) - \lambda F(s) \preceq F(s) - F(\gamma) \preceq F(s),$$

i.e.

$$\frac{s-\gamma}{t-\gamma}F(t) = \frac{\lambda}{1+\lambda}F(t) \preceq F(s)$$

which states that the map

$$(13) \qquad (\gamma, \beta) \ni s \longrightarrow \frac{F(s)}{s-\gamma} \in X$$

is monotonically decreasing.

Fix arbitrarily a strictly increasing sequence $(x_n)_{n\in\mathbb{N}}$ with elements in (γ, β) and such that $x_n \to \beta$ as $n \to \infty$. Then $\left(\frac{F(x_n)}{x_n-\gamma}\right)_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative elements of X. On account of the (assumed) regularity of the positive cone in X, such a sequence is convergent in norm. Let

$$\lim_{n\to\infty} \frac{F(x_n)}{x_n-\gamma} =: b \in X.$$

Observe that b does not depend upon the choice of $(x_n)_{n\in\mathbb{N}}$. Actually, assume that $(\overline{x}_n)_{n\in\mathbb{N}}$ is another strictly increasing sequence of elements from (γ, β), converging to β. Then $\overline{b} := \lim_{n\to\infty} \frac{F(\overline{x}_n)}{\overline{x}_n - \gamma}$ is well defined. Suppose that $b \neq \overline{b}$ and put

$$y_1 := x_1, \quad y_2 := \overline{x}_{n_1}$$

where $n_1 := \min\{n \in \mathbb{N} : \overline{x}_n > x_1\}$, $y_3 := x_{n_2}$, where $n_2 := \min\{n \in \in \mathbb{N} : x_n > \overline{x}_{n_1}\}$, and so on. A sequence $(y_n)_{n\in\mathbb{N}}$ constructed in that way is strictly increasing and tends to β, which implies the convergence of the sequence $\left(\frac{F(y_n)}{y_n}\right)_{n\in\mathbb{N}}$ in X. Obviously, this is impossible because such a sequence contains two subsequences tending to b and $\overline{b}$, respectively.

Consequently, function (13) has a limit $b \in X$ at the point β, which implies that F itself has a limit at β (equal to $(\beta - \gamma)b$) and finishes the proof. ◊

Remark 3. As a matter of fact, the proof above (point (c)) gives more, namely (under the assumption that the positive cone in X is regular) we have:

(a) *if F is concave and bounded below in a neighbourhood of β, then F is continuously extendable to a concave map on $(\alpha, \beta]$;*

(b) *if F is concave and bounded below in a neighbourhood of α, then F is continuously extendable to a concave map on $[\alpha, \beta)$.*

Proposition 1. *Let $(X, \preceq)$ be a vector lattice and let $\alpha \in \mathbb{R}$ and $\beta \in \in \mathbb{R} \cup \{\infty\}$ be given such that $\alpha < \beta$. If $F : (\alpha, \beta) \to X$ is an ε-convex function for some $\varepsilon \succeq 0$, admitting at least two comparable values, then F is order bounded below in a neighbourhood of α.*

Proof. Let $x_0, y_0 \in (\alpha, \beta), x_0 \neq y_0$, be such that $F(x_0)$ and $F(y_0)$ are comparable. Without loss of generality we may assume that $F(x_0) \preceq \preceq F(y_0)$. Suppose that $x_0 < y_0$ and choose arbitrarily an $x \in (\alpha, x_0]$. Then there exists a $\lambda(x) \in (0, 1]$ such that $x_0 = \lambda(x)x + (1 - \lambda(x))y_0$ whence

$$F(x_0) \preceq \lambda(x)F(x) + (1 - \lambda(x))F(y_0) + \varepsilon$$

or, equivalently,

$$a := F(x_0) - F(y_0) - \varepsilon \preceq \lambda(x)[F(x) - F(y_0)].$$

Consequently, because of $F(x_0) - F(y_0) \preceq 0 \preceq \varepsilon$, we infer that $a \preceq 0$ and

$$m := \frac{y_0 - \alpha}{y_0 - x_0} a + F(y_0) \preceq \frac{1}{\lambda(x)} a + F(y_0) \preceq F(x).$$

Since, evidently, m does not depend upon x, we deduce that the restriction $F|_{(\alpha, x_0]}$ is bounded below by m.

Suppose now that $y_0 < x_0$ and choose arbitrarily an $x \in (\alpha, y_0]$. Then there exists a $\lambda(x) \in (0,1]$ such that $y_0 = \lambda(x)x + (1-\lambda(x))x_0$ whence

$$F(y_0) \preceq \lambda(x)F(x) + (1-\lambda(x))F(x_0) + \varepsilon$$

which implies that

$$-\varepsilon + \lambda(x)F(x_0) \preceq F(x_0) - F(y_0) - \varepsilon + \lambda(x)F(x) \preceq \lambda(x)F(x),$$

i.e.

$$F(x) \succeq F(x_0) - \frac{1}{\lambda(x)}\varepsilon = F(x_0) + \frac{x_0 - x}{y_0 - x_0}\varepsilon \succeq F(x_0) + \frac{x_0 - \alpha}{y_0 - x_0}\varepsilon =: \tilde{m},$$

which states that the restriction $F|_{(\alpha, x_0]}$ is bounded below by $\tilde{m}$ and finishes the proof. ◊

As a corollary, we get easily the following

Theorem 6. *Let $(X, \|\cdot\|, \preceq)$ be a Banach lattice and let $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, be given. If the positive cone in X is regular and $F : (\alpha, \beta) \to X$ is a concave and ε-convex map admitting at least two comparable values, then there exists an affine function $a : (\alpha, \beta) \to X$ that separates F and $F + \varepsilon$.*

Proof. In view of Prop. 1, F is order bounded below in a neighbourhood of α. An appeal to Th. 5 assures that F admits a continuous extension to a map $\tilde{F} : [\alpha, \beta) \to X$ which, obviously, is both concave and ε-convex. To finish the proof it remains to apply Th. 3 for the extended map $\tilde{F}$. ◊

The local boundedness below at the left endpoint of the domain played here the crucial role. In the next proposition we shall derive that property of ε-convex mappings on bounded open interval from an additional assumption upon the positive cone in the target space.

Proposition 2. *Let $(X, \|\cdot\|, \preceq)$ be a Banach lattice and let $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, be given. If ε is an inner point of the positive cone (in the sense of norm topology) and $F : (\alpha, \beta) \to X$ is a continuous ε-convex mapping, then F is locally order bounded below at α.*

Proof. Write $C := \{u \in X : u \succeq 0\}$ and fix arbitrarily an $x_0 \in (\alpha, \beta)$. Since, by assumption, $\varepsilon \in \operatorname{int} C$, the set $\varepsilon - C$ yields a neighbourhood of zero. Thus, by continuity, we may find a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (\alpha, \beta)$ and $F((x_0 - \delta, x_0 + \delta)) \subset F(x_0) + \varepsilon - C$. Take a $y_0 \in (x_0, x_0 + \delta)$; then $a := F(y_0) - F(x_0) - \varepsilon \preceq 0$. Now, for every $x \in (\alpha, x_0)$ one has $x_0 = \lambda(x)x + (1-\lambda(x))y_0$ with $\lambda(x) := \frac{y_0 - x_0}{y_0 - x} \in (0,1)$ and $\lambda(x) > \frac{y_0 - x_0}{y_0 - \alpha} =: \lambda_0 \in (0,1)$. On the other hand,

$$F(x_0) \preceq \lambda(x)F(x) + (1-\lambda(x))F(y_0) + \varepsilon, \quad x \in (\alpha, x_0),$$

whence

$$\frac{1}{\lambda_0}a \preceq \frac{1}{\lambda(x)}a \preceq F(x) - F(y_0), \quad x \in (\alpha, \gamma],$$

which means that $F|_{(\alpha,\gamma]}$ is order bounded below by $F(y_0) + \frac{1}{\lambda_0}a$ and completes the proof. ◊

Applying Prop. 2 instead of Prop. 1 and recalling that a concave mapping on an open interval is continuous (see Cor. 4) we obtain the following

Theorem 7. *Let $(X, \|\cdot\|, \preceq)$ be a Banach lattice and let $\alpha, \beta \in \mathbb{R}, \alpha < < \beta$, be given. If the positive cone $C \subset X$ is regular and if $F : (\alpha, \beta) \to \to X$ is a concave and ε-convex mapping with $\varepsilon \in \operatorname{int} C$, then there exists an affine function $a : (\alpha, \beta) \to X$ that separates F and $F + \varepsilon$.*

4. Two examples

In connection with Th. 6 it seems worthy to observe that for every Banach space $(X, \|\cdot\|)$ of dimension at least two and any nondegenerate interval $I \subset \mathbb{R}$ one may construct a regular cone $C \subset X$ and a concave map $F : I \to X$ being ε-convex for each $\varepsilon \in C$ such that none of two different values of F are comparable. Actually, fix arbitrarily two linearly independent vectors c, d from X and put

$$F(x) := xc, \quad x \in I, \quad C := \{\lambda d : \lambda \geq 0\}.$$

Clearly, F is both concave and convex (and hence ε-convex for every $\varepsilon \in \in C$) but for each pair (x, y) of distinct elements from the interval I one has $F(x) - F(y) = (x - y)c \notin C$. The cone C itself is regular because given an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of elements of X majorized by an $m \in X$ we derive easily the existence of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of nonnegative reals such that

$$u_{n+1} = u_1 + (\lambda_1 + \cdots + \lambda_n)c \quad \text{for all} \quad n \in \mathbb{N}.$$

Since we have also $m - u_{n+1} \in C$ for all $n \in \mathbb{N}$, we infer that

$$m - u_1 = \left(\alpha_n + \sum_{k=1}^{n} \lambda_k\right) c \quad \text{for all} \quad n \in \mathbb{N},$$

with some nonnegative α_n's. Now, for every $n \in \mathbb{N}$, we get

$$0 \leq \sum_{k=1}^{n} \lambda_k \leq \alpha_n + \sum_{k=1}^{n} \lambda_k = \frac{\|m - u_1\|}{\|c\|}$$

which proves that the series $\left(\sum_{k=1}^{n} \lambda_k\right)_{n\in\mathbb{N}}$ is convergent. Consequently, the sequence $(u_n)_{n\in\mathbb{N}}$ tends (in norm) to $u_1 + \left(\sum_{k=1}^{\infty} \lambda_k\right) c$.

Our next example shows that the regularity assumption upon the positive cone in Th. 6 is not a necessary condition for the separation effect desired. To visualize it, consider a Banach lattice $C([0,1],\mathbb{R})$ of all continuous real functions on $[0,1]$ with the uniform convergence norm and the natural (pointwise) ordering relation. The positive cone P here has nonempty interior (each member of $C([0,1],\mathbb{R})$ with positive absolute minimum is an inner point) but that lattice is not boundedly complete (see e.g. H.–U. Schwarz [11]); in particular, P is not regular. Nevertheless, a map $F:(0,1)\to C([0,1],\mathbb{R})$ given by the formula

$$F(x)(s) := x^s, \quad x\in(0,1), s\in[0,1],$$

which is concave and ε-convex with

$$\varepsilon(s) := \begin{cases} 1 & \text{for} \quad s=0 \\ s^{\frac{s}{1-s}} - s^{\frac{1}{1-s}} & \text{for} \quad s\in(0,1) \\ 0 & \text{for} \quad s=1 \end{cases}$$

admits an affine transformation that separates F and $F+\varepsilon$; namely a map $a:(0,1)\to C([0,1],\mathbb{R})$ given by the formula $a(x):=x+\varepsilon, x\in$ $\in (0,1)$, will do (obvious). We omit the detailed quite elementary calculations showing the concavity and ε-convexity of F and the fact that our choice of ε is sharp in the sense that it is the smallest possible continuous function on $[0,1]$ among the totality of functions $\delta\in P$ for which F is δ-convex.

5. Concluding remarks

Note that each concave function F from a bounded real interval $I\subset\mathbb{R}$ into $\mathbb{R}^n$ (understood as Banach lattice with the order relation $\preceq$ generated by the cone C consisting of vectors having all coordinates nonnegative), admitting a continuous extension onto the closure of I, is automatically ε-convex with some $\varepsilon\in C$. In fact, let $\tilde{F}:\operatorname{cl} I\to\mathbb{R}^n$ be a continuous extension of a concave mapping $F:I\to\mathbb{R}^n$ onto the closure of I. Put $K:=\operatorname{cl} I\times\operatorname{cl} I\times[0,1]$ and define a transformation $T=(T_1,\dots,T_n):K\to\mathbb{R}^n$ with the aid of the formula

$$T(x,y,\lambda) := \tilde{F}(\lambda x+(1-\lambda)y) - \lambda\tilde{F}(x) - (1-\lambda)\tilde{F}(y),$$

for $x,y\in\operatorname{cl} I$ and $\lambda\in[0,1]$. Since all the T_i's are continuous on a compact domain, for each $i\in\{1,\dots,n\}$ we have

$$\mu_i := \sup\{T_i(x,y,\lambda):(x,y,\lambda)\in K\} < \infty.$$

Setting $\varepsilon_i := \max(0,\mu_i), i\in\{1,\dots,n\}$, one can easily check that $\tilde{F}$, and hence also F itself, is ε-convex with $\varepsilon := (\varepsilon_1,\dots,\varepsilon_n)\in C$.

More generally, each concave function F from a bounded real interval $I \subset \mathbb{R}$ into a Banach lattice $(X, \|\cdot\|, \preceq)$ with a strong unit, admitting a continuous extension onto the closure of I, is ε-convex with some nonnegative ε. Actually, F is continuous in $\operatorname{int} I$ (cf. Remark 1) and if $\tilde{F} : \operatorname{cl} I \to X$ stands for a continuous extension of F onto $\operatorname{cl} I$, then $T(K)$ is compact and *a fortiori* norm bounded in X. In a Banach lattice with a strong unit each norm bounded set is also order bounded (see G. Birkhoff [2, p. 472]).

Any affine map a from a nondegenerate interval $I \subset \mathbb{R}$ into a real Banach space $(X, \|\cdot\|)$ is of the form

$$a(x) = x \cdot c + d, \quad x \in I,$$

where c, d are some fixed vectors from X. Indeed, obviously, a satisfies the Jensen functional equation

$$a\left(\frac{x+y}{2}\right) = \frac{a(x) + a(y)}{2}, \quad x, y \in I,$$

and a is norm bounded on each compact subinterval $[\alpha, \beta]$ of the interval I, because for every $x \in [\alpha, \beta]$ there exists a $\lambda \in [0,1]$ with $x = \lambda\alpha + (1-\lambda)\beta$ whence

$$\|a(x)\| = \|\lambda a(\alpha) + (1-\lambda)a(\beta)\| \leq \max\left(\|a(\alpha)\|, \|a(\beta)\|\right).$$

Therefore, in view of [5, Th. 1], there exists an additive map $A : \mathbb{R} \to X$ and a vector $d \in X$ such that $a(x) = A(x) + d, x \in I$. Clearly, $A|_{[\alpha,\beta]}$ is norm bounded and hence continuous. In particular, A is linear whence

$$A(x) = x \cdot A(1), \quad x \in \mathbb{R},$$

and it remains to put $c := A(1)$.

The sandwich type theorems presented in the present paper may also be viewed as (partial) generalizations of the stability result for affine functions established by K. Nikodem and Sz. Wąsowicz [9] in the class of real functions on a real interval I. They have proved that any function $f : I \to \mathbb{R}$ satisfying the inequality

$$|f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda)f(y)| \leq \varepsilon, \quad x, y \in I, \lambda \in [0,1]$$

differs from an affine function by at most $\frac{1}{2}\varepsilon$ in absolute value. Our results concern the case where a given map F from I into a vector (resp. Banach) lattice X satisfies the inequality

$$0 \preceq F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y) \preceq \varepsilon, \quad x, y \in I, \lambda \in [0,1],$$

which implies both

$$|F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y)| \preceq \varepsilon, \quad x, y \in I, \lambda \in [0,1],$$

and

$$\|F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y)\| \leq \|\varepsilon\|, \quad x, y \in I, \lambda \in [0,1]$$

(the latter one in the Banach lattice case). Our claim is that then (under suitable assumptions trivially satisfied in the real case) there exist vectors $c, d \in X$ such that

$$0 \preceq x \cdot c + d - F(x) \preceq \varepsilon \quad \text{for all} \quad x \in I,$$

which in turn implies that

$$|x \cdot c + d - F(x)| \preceq \varepsilon \quad \text{for all} \quad x \in I,$$

and

$$\|x \cdot c + d - F(x)\| \leq \|\varepsilon\| \quad \text{for all} \quad x \in I$$

(the latter one in the Banach lattice case). However, the result of K. Nikodem and Sz. Wąsowicz is not covered completely because we were dealing with concave mappings and they were admitting ε-concave ones. Nevertheless, a thorough inspection of the proofs shows that in some cases (which we do not enumerate here explicitly) such an improvement of our statements is possible without essential proof changes.

References

[1] BARON, K., NIKODEM K. and MATKOWSKI, J.: A sandwich with convexity, *Mathematica Pannonica* **5**/1 (1994), 139–144.

[2] BIRKHOFF, G.: Lattice Theory, *3-rd edition, American Mathematical Society, Colloquium Publications* **25**, *Providence, Rhode Island, 1967.*

[3] CHOLEWA, P. W.: Remarks on the stability of functional equations, *Aequationes Mathematicae* **27** (1984), 76–86.

[4] GER, R.: Convex transformations with Banach lattice range, *Stochastica* **11** (1987), 13–23.

[5] GER, R.: On extensions of polynomial functions, *Results in Mathematics* **26** (1994), 281–289.

[6] HYERS, D. H. and ULAM, S. M.: Approximately convex functions, *Proceedings of the American Mathematical Society* **3** (1952), 821–828.

[7] KRASNOSELSKIJ, M. A.: Regular and completely regular cones [Russian], *Doklady Akad. Nauk SSSR* **135** (1960), 255–257.

[8] KUCZMA, M.: An Introduction to the Theory of Functional Equations and Inequalities, Polish Scientific Publishers and Silesian University, Warsaw-Cracow-Katowice, 1985.

[9] NIKODEM, K. and WĄSOWICZ Sz.: A sandwich theorem and Hyers–Ulam stability of affine functions, *Aequationes Mathematicae* **49** (1995), 160–164.

[10] RODÉ, G.: Eine abstrakte Version des Satzes von Hahn–Banach, *Archiv der Mathematik* **31** (1978), 474–481.

[11] SCHWARZ, H.–U.: Banach Lattices and Operators, Teubner–Texte zur Mathematik **71**, Leipzig, 1984.