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ON THE DEGREE OF NILPOTENCY OF THE RADICAL OF RELATIVELY FREE ALGEBRAS

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Abstract: We show that the index of nilpotency of the Jacobson radical of a relatively free algebra of finite rank in a variety of associative algebras over a field of characteristic zero defined by a T-ideal I is bounded by a constant multiple of the rank, where the constant depends only on I. For T-semiprime I we express such a constant as a function of the PI-degree.

1. Introduction

We work with unitary associative algebras over a fixed field K of

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characteristic zero. Let $K\langle x_1, x_2, \ldots\rangle$ be the free associative algebra of countable rank. An ideal I of the free algebra is called a *T-ideal,* if I is closed under K-algebra endomorphisms. For any K-algebra R denote by

$$T(R) = \{f \in K\langle x_1, x_2, \ldots\rangle \mid f = 0 \text{ is a polynomial identity on } R\}$$

the *T-ideal of identities* of R.

The T-ideals of identities of the $n \times n$ matrix algebras $M_n = M_n(K)$ over K ($n = 1, 2, \ldots$) play a special role among T-ideals. For any T-ideal I denote by $n(I)$ the maximal natural number n such that I is contained in $T(M_n)$, this number is called the *PI-degree* of I. By an old theorem of Amitsur in [1] if f is contained in $T(M_{n(I)})$ then I contains some power of f. Moreover, we can say more if we restrict the number of variables. For any T-ideal I denote by I_m the intersection of I with the free algebra $K\langle x_1, \ldots, x_m\rangle$ of rank m. By the Razmyslov-Kemer-Braun Theorem ([9], [7], [4]) we know that $T_m(M_{n(I)})$ is nilpotent modulo I_m, say of index $d_m(I)$. In general $T(M_{n(I)})$ is not nilpotent modulo I, which implies that $d_m(I)$ goes to infinity as m grows. Clearly, the number $d_m(I)$ is a function of I and m. In this paper we show that $d_m(I) \leq Cm$, where C is some constant depending only on I. Moreover, we prove that for a T-semiprime I we may take $C = 2n^2(I) + 1$.

2. Preliminaries

Kemer developed a structure theory for T-ideals in [6]. We recall his results. Any T-ideal I is contained in a unique minimal T-semiprime T-ideal S. S is nilpotent modulo I. Any T-semiprime T-ideal is the intersection of finitely many T-prime T-ideals. The T-prime T-ideals are the following: $T(M_n)$, $T(M_n(G))$ and $T(M_{n,k})$ $(n, k \in \mathbb{N}, n > k \geq n/2)$, where

$$G = K\langle v_1, v_2, \ldots \mid v_iv_j + v_jv_i = 0 \quad i,j = 1, 2, \ldots\rangle$$

denotes the infinite dimensional Grassmann algebra, $M_n(G)$ is the $n \times n$ matrix algebra over G, and $M_{n,k}$ denotes the $\mathbb{Z}_2$-graded subalgebra of $M_n(G)$ consisting of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that A (D, resp.) is a $k \times k$ ($(n-k) \times (n-k)$, resp.) matrix over G_0, and B (C, resp.) is an $k \times (n-k)$ ($(n-k) \times k$, resp.) matrix over G_1. ($G = G_0 + G_1$ is the usual $\mathbb{Z}_2$-grading on G, that is, G_0 (G_1, resp.) is spanned by monomials of even (odd, resp.) length in the generators v_i.) Though we have some information about the cocharacter series of $T(M_n(G))$ and $T(M_{n,k})$ (see for example [2], [3]), we know only very little identities explicitly. However, we have $n(T(M_k(G))) = k$ and $n(T(M_{k,l})) = l$ where $k > l \geq k/2$ (see [5]). Here we give explicit bounds on $d_m(T(M_n(G))$ and $d_m(T(M_{n,k}))$. In particular, we get many explicit identities on the algebras $M_n(G)$ and $M_{n,k}$.

Proposition 2.1. *We have*

$$d_m(T(M_n(G)) \leq \frac{1}{2}n^2m + 1.$$

In other words, let $f_1, \ldots, f_d \in K\langle x_1, \ldots, x_m\rangle$ be elements of the T-ideal of identities of M_n. If $d > \frac{1}{2}n^2m$, then $f_1 \ldots f_d = 0$ is an identity on $M_n(G)$.

Proof. Denote by

$$U(r) = (x_{ij}(r)) \quad (r = 1, \ldots, m)$$

$n \times n$ matrices whose entries are non-commuting indeterminates. We may substitute $U(1), \ldots, U(m)$ into the polynomials $f_1, \ldots, f_d$, and we get matrices

$$f_s(U(1), \ldots, U(m)) = (f_{sij}) \quad (s = 1, \ldots, d),$$

where the entries f_{sij} are elements of the free algebra

$$K\langle x_{ij}(r) | 1 \leq i,j \leq n, \quad 1 \leq r \leq m\rangle$$

The assumption that $f_s = 0$ is a polynomial identity on $M_n(K)$ implies that each f_{sij} is contained in the commutator ideal of the free algebra. Hence f_{sij} can be written as a linear combination of polynomials of the form

$$(*) \qquad z_1 \ldots z_k[z_{k+1}, z_{k+2}]z_{k+3} \ldots z_u,$$

where $z_1, \ldots, z_u \in \{x_{ij}(r) | 1 \leq i,j \leq n, \quad 1 \leq r \leq m\}$ and $[z, w] = zw - wz$ denotes the commutator of z and w. Now the entries of $(f_1 \ldots f_d)(U(1), \ldots, U(m))$ are linear combinations of products of d polynomials of type $(*)$. For each such product we have a variable $z \in \{x_{ij}(r) | 1 \leq i,j \leq n, \quad 1 \leq r \leq m\}$ occurring in at least two commutators by the inequality $d > \frac{1}{2}n^2m$. So it suffices to show that $f[z,u]g[z,v]h$ is contained in $T(G)$ for any f, g, h, u, v, z. $T(G)$ is generated by $[[x_1, x_2], x_3]$ (c.f. [8]), and any commutator is central modulo this identity. Hence the claim follows from the well known fact that $[z, u][z, v] = 0$ is an identity on G. $\Diamond$

The bound in Prop. 2.1 is sharp in case $n = 1$. For example,

$$[x_1, x_2] + [x_3, x_4] + \ldots + [x_{2d-1}, x_{2d}] = 0$$

is an identity on K, and

$$([v_1, v_2] + [v_3, v_4] + \ldots + [v_{2d-1}, v_{2d}])^d = 2d! v_1 \ldots v_{2d} \neq 0,$$

where $v_1, \ldots, v_{2d}$ are generators of G.

Apply our result to $M_2(G)$ and the standard polynomial

$$S_4(x_1, x_2, x_3, x_4) = \sum_{\pi \in Sym(4)} \operatorname{sign}(\pi) x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} x_{\pi(4)}.$$

We get that $S_4^9 = 0$ is an identity on $M_2(G)$. However, this is not the best possible result. The cocharacter series of $T(M_2(G))$ is contained in the $(4,4)$ hook (see [3]). Since S_5^5 generates an irreducible Gl_5–module corresponding to the partition $(5,5,5,5,5)$, it follows that $S_5^5 = 0$ is an identity on $M_2(G)$. On substituting x_5 by 1 we obtain that $S_4^5 = 0$ is an identity on $M_2(G)$. We can get also that $[[x,y]^2, x]^5 = 0$ is an identity on $M_2(G)$.

Proposition 2.2. *For any positive integers $n > k \geq n/2$ we have*

$$d_m(T(M_{n,k})) \leq 2mk(n-k) + 1.$$

In other words, let $f_1, \ldots, f_d \in K\langle x_1, \ldots, x_m \rangle$ be elements of the T-ideal of identities of M_k. If $d > 2mk(n-k)$, then $f_1 \ldots f_d = 0$ is a polynomial identity on $M_{n,k}$.

Proof. We change slightly the notation of the proof of the previous proposition. Put

$$U(r) = \begin{pmatrix} U_{11}(r) & U_{12}(r) \\ U_{21}(r) & U_{22}(r) \end{pmatrix} \quad (r = 1, \ldots, m)$$

where $U_{11}(r) = (y_{ij}(r))$ is a $k \times k$ matrix, $U_{22}(r) = (y'_{ij}(r))$ is an $(n-k) \times (n-k)$ matrix, $U_{12}(r) = (z_{ij}(r))$ is a $k \times (n-k)$ matrix and $U_{21}(r) = (z'_{ij}(r))$ is an $(n-k) \times k$ matrix, and the entries y, y', z, z' are non-commuting indeterminates. It is easy to see that

$$(**) \quad f_s(U(1), \ldots, U(m)) =$$

$$= \begin{pmatrix} f_s(U_{11}(1), \ldots, U_{11}(m)) & 0 \\ 0 & f_s(U_{22}(1), \ldots, U_{22}(m)) \end{pmatrix} + A_s,$$

where each monomial of each entry of A_s contains a variable z or z'. We have to show that $(f_1 \ldots f_d)(U(1), \ldots, U(m))$ vanishes whenever we substitute the variables y, y' by elements of G_0 and the variables z, z' by elements of G_1. $f_s = 0$ is a polynomial identity on M_k (and on M_{n-k}), and G_0 is commutative, so the first summand of the right hand side of $(**)$ vanishes under such a substitution. So it suffices to show that $A_1 \ldots A_d$ vanishes. The inequality $d > 2mk(n-k)$ implies that for each monomial of each entry of $A_1 \ldots A_d$ there exists a variable z or z' which occurs at least twice. Now G_0 is the center of G, and the elements of G_1 anticommute. Moreover, we have $x^2 = 0$ for any $x \in G_1$, so such monomials vanish under the prescribed substitutions, showing the claim. $\Diamond$

For example, apply the proposition for the algebra $M_{2,1}$ (which has the same identities as $G \otimes G$) and the polynomial $[x, y]$. We get that $[x, y]^5 = 0$ is an identity on $M_{2,1}$.

3. Main results

For any T-ideal I denote by $s(I)$ the smallest positive integer s such that S^s is contained in I, where S is the minimal T-semiprime T-ideal containing I.

Theorem 3.1. *For any T-ideal I we have $d_m(I) \leq s(I)(2mn^2(I) + 1)$.*

Proof. Let S be the minimal T-semiprime T-ideal containing I. By the results of Kemer quoted in Section 2 we have $S = P_1 \cap \ldots \cap P_t$, where $P_1, \ldots, P_t$ are T-prime T-ideals. Now $n(I) = n(S) = \max\{n(P_i) | i = 1, \ldots, t\}$ (c.f. [5]). Hence if $d \geq s(I)(2mn^2(I) + 1)$ and $f_1, \ldots, f_d$ are $m-$variable polynomial identities of $M_{n(I)}$, then by Prop. 2.1 and 2.2 $f_1 \cdots f_d \in S^{s(I)} \subseteq I$. $\Diamond$

The next statement shows that we can not expect better general upper bound for the order of magnitude of $d_m(I)$ as a function of m.

Proposition 3.2. *If the T-ideal I does not contain any power of $T(M_{n(I)})$, then $d_m(I) > \frac{1}{2n(I)} m$.*

Proof. The condition on I means that $T(M_k(G))$ or $T(M_{k,l})$ (for some k, l) occurs among $P_1, \ldots, P_t$ (they are the T-prime T-ideals of I as in the proof above). Consider d copies of the standard polynomial $S_{2n(I)}$ in pairwise disjoint sets of variables. The product of them is clearly not an identity on any $M_k(G)$ or on any $M_{k,l}$ (since it is not an identity on G), hence it is not contained in I. This shows the required inequality, since $S_{2n(I)} \in T(M_{n(I)})$. $\Diamond$

Remark. Obviously, if I contains some power of $T(M_{n(I)})$ (or equivalently, I contains some standard polynomial), then $d_m(I)$ is bounded as a function of m.

For any T-ideal I and integer m the Jacobson radical of the relatively free algebra $K\langle x_1, \ldots, x_m\rangle / I_m$ is $T_m(M_{n(I)})/I_m$. Hence we may express the content of our theorem in a slightly different way.

Corollary 3.3. *Let R be a relatively free algebra of rank m in a* *variety of unitary K-algebras. There exists a constant C depending on the variety such that the index of nilpotency of the Jacobson radical of R is at most Cm. Moreover, if the T-ideal of identities of the variety is T-semiprime, then we may take $C = 2n^2(T(R)) + 1$.* $\diamond$

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