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LINKED DARBOUX MOTIONS

Otto **Röschel**

Institute of Geometry, TU Graz, Kopernikusgasse 24, A-8010 Graz, Austria

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Abstract: Given two Darboux motions $\Sigma_1 \setminus \Sigma_0$ and $\Sigma_2 \setminus \Sigma_0$ in Euclidean 3-space, which are linked by common angle of rotation as time parameter. Then we are able to show, that the relative motion $\Sigma_2 \setminus \Sigma_1$ is line-symmetric in the sense of J. Krames with a special ruled surface of degree 4 as basic surface. Further in Σ_2 in the general case there exists at least a two-parametric family of real points, which are moved on spheres under $\Sigma_2 \setminus \Sigma_1$. Examples finish the paper.

0. In the last years a great variety of overconstrained linkages has been studied, which are gained by linking systems with spherical $2R$-links (see [14], [15], [17], [18], [19], [20], [21]). The most famous of these linkages seem to be the socalled Heureka-polyhedron (see [18], [21] and the references given there). It has been shown, that most of these models consist of relative motions gained by combining an axial Darboux motion with an inverse of another (in most cases congruent) Darboux motion — both parametrized with respect to their angle of rotation. In generalisation to these facts we here start with not necessarily axial Darboux motions.

1. In the 3-dimensional Euclidean space E_3 we use cartesian frames $\{O_i; x_i, y_i, z_i\}$ $(i = 0, 1, 2)$ to describe points of given systems Σ_i $(i = = 0, 1, 2)$ by their position vectors. There Σ_1 and Σ_2 are moved by

Darboux motions $\Sigma_1 \setminus \Sigma_0$ and $\Sigma_2 \setminus \Sigma_0$ with respect to the fixed frame Σ_0. Both one-parameter-motions shall be parametrized by the angle of rotation denoted by t. We will call such Darboux motions *linked Darboux motions*. The Darboux motions are cylindrical and therefore have fixed directions. In Σ_1 (Σ_2) the z_1- (z_2-)axis shall denote this fixed direction. Then a parametrization of such a motion may be given by

$$\text{(1)}\quad \begin{aligned}\vec{x}_0(t, \vec{x}_i) :=& \begin{pmatrix} a_i^* \\ c_i^* \\ e_i^* + z_i \end{pmatrix} + \begin{pmatrix} -a_i^* + x_i \\ -c_i^* + y_i \\ -e_i^* \end{pmatrix} \cos t + \\ &+ \begin{pmatrix} b_i^* - y_i \\ d_i^* + x_i \\ f_i^* \end{pmatrix} \sin t \qquad (t \in [0, 2\pi])\end{aligned}$$

with real constants $a_i^*, \ldots, f_i^*$ ($i = 1, 2$; cf. [1, p. 306]). Rotation about the x_0-axis through angle φ_i gives representations of $\Sigma_1 \setminus \Sigma_0$ and $\Sigma_2 \setminus \Sigma_0$ in the form

$$\text{(2)}\quad \begin{aligned}\vec{x}_0(t, \vec{x}_i) :=& \begin{pmatrix} a_i \\ c_i - z_i \sin\varphi_i \\ e_i + z_i \cos\varphi_i \end{pmatrix} + \begin{pmatrix} -a_i + x_i \\ -c_i + y_i \cos\varphi_i \\ -e_i + y_i \sin\varphi_i \end{pmatrix} \cos t + \\ &+ \begin{pmatrix} b_i - y_i \\ d_i + x_i \cos\varphi_i \\ f_i + x_i \sin\varphi_i \end{pmatrix} \sin t \qquad (t \in [0, 2\pi])\end{aligned}$$

($i = 1, 2$) with other real constants $a_i := a_i^*$, $b_i := b_i^*$, $c_i := c_i^* \cos\varphi_i - e_i^* \sin\varphi_i$, $d_i := d_i^* \cos\varphi_i - f_i^* \sin\varphi_i$, $e_i := c_i^* \sin\varphi_i + e_i^* \cos\varphi_i$, $f_i := d_i^* \sin\varphi_i + f_i^* \cos\varphi_i$. In Σ_0 the z_0-axis shall be a line of symmetry for the two fixed directions $(0, -\sin\varphi, \cos\varphi)$ and $(0, \sin\varphi, \cos\varphi)$. Therefore we have $\varphi = -\varphi_1 = \varphi_2$.

2. The relative motion $\Sigma_2 \setminus \Sigma_1$. By taking the inverse motion of $\Sigma_1 \setminus \Sigma_0$ and then combining it with $\Sigma_2 \setminus \Sigma_0$ we get the following representation of the point paths of $\Sigma_2 \setminus \Sigma_1$

$$\text{(3)}\quad \begin{aligned}&\vec{x}_1(t, \vec{x}_2) = \\ &= \begin{pmatrix} A\cos t + C\sin t\cos\varphi - E\sin t\sin\varphi \\ -A\sin t + C\cos t\cos\varphi - E\cos t\sin\varphi \\ C\sin\varphi + E\cos\varphi \end{pmatrix} (1 - \cos t) + \\ &+ \begin{pmatrix} B\cos t + D\sin t\cos\varphi - F\sin t\sin\varphi \\ -B\sin t + D\cos t\cos\varphi - F\cos t\sin\varphi \\ D\sin\varphi + F\cos\varphi \end{pmatrix} \sin t + \\ &+ \begin{pmatrix} \cos^2 t + \sin^2 t\cos 2\varphi & -\sin t\cos t(1 - \cos 2\varphi) & -\sin t\sin 2\varphi \\ -\sin t\cos t(1 - \cos 2\varphi) & \sin^2 t + \cos^2 t\cos 2\varphi & -\cos t\sin 2\varphi \\ \sin t\sin 2\varphi & \cos t\sin 2\varphi & \cos 2\varphi \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}.\end{aligned}$$

There we have used abbreviations $A := a_2 - a_1, \ldots, F := f_2 - f_1$. By substituting $u := \tan t/2$ we see, that this motion has a rational parametrisation of degree 4. It is a special motion of those considered by the author in [13] — there motions with this property were called "rational one-parameter motions of degree 4". Therefore we have

Theorem 1. *The relative motion* $\Sigma_2 \backslash \Sigma_1$ *of two linked Darboux motions* $\Sigma_1 \backslash \Sigma_0$ *and* $\Sigma_2 \backslash \Sigma_0$ *(in general) is a special rational motion of degree* 4*. All point paths allow a rational parametrisation of degree* 4 *by a common parameter.*

Remark. Formula (3) shows that our relative motion $\Sigma_2 \backslash \Sigma_1$ does not change its representation, if we fix the constants $A, B, \dots, F$ and change our input data $a_i, \dots, f_i$ according to the fixed values $A = a_2 - a_1, \dots, F = f_2 - f_1$. So we are allowed to take a very special Darboux motion to be the first generating motion $\Sigma_1 \backslash \Sigma_0$. The second generating Darboux motion then has to be constructed as mentioned above.

3. The relative motion $\Sigma_2 \backslash \Sigma_1$ is line-symmetric in the sense of J. Krames. We now are able to prove the following

Theorem 2. *The relative motion* $\Sigma_2 \backslash \Sigma_1$ *of two linked Darboux motions* $\Sigma_1 \backslash \Sigma_0$ *and* $\Sigma_2 \backslash \Sigma_0$ *(in general) is a special line-symmetric motion of degree* 4 *in the sense of J. Krames* [5]–[11]*. The basic surface of these line-symmetric motions is generated as path of a straight line under the inverse motion* $\Sigma_0 \backslash \Sigma_1$ *or* $\Sigma_0 \backslash \Sigma_2$ *of one of the generating Darboux motions.*

These relative motions already have been studied by R. Bricard [3] and J. Krames [10].

Proof. A) In [13] I proved that rational motions of degree 4 may have real points with plane paths. We look for these points now: We determine

$$
\begin{aligned}
(4)\quad & \vec{x}_1(t, \vec{x}_2) = \\
&= \begin{pmatrix} -A \\ -C\cos\varphi + E\sin\varphi + z_2\sin 2\varphi \\ C\sin\varphi + E\cos\varphi - y_2\sin 2\varphi \end{pmatrix} \sin t + \begin{pmatrix} C\cos\varphi - E\sin\varphi - z_2\sin 2\varphi \\ -A \\ D\sin\varphi + F\cos\varphi + x_2\sin 2\varphi \end{pmatrix} \cos t + \\
&+ \begin{pmatrix} A + D\cos\varphi - F\sin\varphi - x_2(1-\cos 2\varphi) \\ -B + C\cos\varphi - E\sin\varphi + y_2(1-\cos 2\varphi) \\ 0 \end{pmatrix} \sin 2t + \\
&+ \begin{pmatrix} B - C\cos\varphi + E\sin\varphi - y_2(1-\cos 2\varphi) \\ A + D\cos\varphi - F\sin\varphi - x_2(1-\cos 2\varphi) \\ 0 \end{pmatrix} \cos 2t.
\end{aligned}
$$

For the special points

$$(5) \qquad g_2 \ldots \qquad \begin{aligned} A + D\cos\varphi - F\sin\varphi - x_2(1-\cos 2\varphi) = 0, \\ B - C\cos\varphi + E\sin\varphi - y_2(1-\cos 2\varphi) = 0 \end{aligned}$$

these derivation vectors are parallel to a plane and therefore the corresponding points of Σ_2 describe plane paths in Σ_1 . If we have $\sin 2\varphi \neq 0$ (the generating Darboux motions then have different fixed directions — our relative motion is not cylindrical; the case $\sin 2\varphi = 0$ will be treated seperatly in chapter 5) we have a straight line g_2 fixed in Σ_2 with plane point paths (these paths in general are ellipses — see (4)). But in the case $\sin 2\varphi \neq 0$ the points of the straight line

$$(6) \qquad h_2 \ldots \qquad \begin{aligned} -D\sin\varphi - F\cos\varphi = x_2 \sin 2\varphi, \\ C\sin\varphi + E\cos\varphi = y_2 \sin 2\varphi \end{aligned}$$

describe plane paths, too — see (4). The planes containing the paths then are parallel. Summing up we at least have *two straight lines* fixed in Σ_2 with *plane point-paths* under the relative motion $\Sigma_2 \setminus \Sigma_1$.

B) The straight line g_2 may be moved into the z_2-axis of Σ_2. Our new origin may be moved onto this axis, too. We put

$$(7) \qquad \begin{aligned} x_2{}^* &:= \big[A + D\cos\varphi - F\sin\varphi - x_2(1-\cos 2\varphi)\big]/(1-\cos 2\varphi), \\ y_2{}^* &:= \big[B - C\cos\varphi + F\sin\varphi - y_2(1-\cos 2\varphi)\big]/(1-\cos 2\varphi), \\ z_2{}^* &:= \big[C\cos\varphi - E\sin\varphi - z_2 \sin 2\varphi)\big]/\sin 2\varphi. \end{aligned}$$

For symmetry reasons we change the position of the origin of the first system, too:

$$(8) \qquad \begin{aligned} x_1{}^* &:= \big[-A - D\cos\varphi - F\sin\varphi - x_1(1-\cos 2\varphi)\big]/(1-\cos 2\varphi), \\ y_1{}^* &:= \big[-B + C\cos\varphi + E\sin\varphi - y_1(1-\cos 2\varphi)\big]/(1-\cos 2\varphi), \\ z_1{}^* &:= \big[C\cos\varphi + E\sin\varphi - z_1 \sin 2\varphi)\big]/\sin 2\varphi. \end{aligned}$$

We substitute according to (7) and (8) and compute the vectors

$$(9) \qquad \begin{aligned} &\vec{d}_0(t, \vec{x}_1{}^*, \vec{x}_2{}^*) := \vec{x}_0(t, \vec{x}_2{}^*) - \vec{x}_0(t, \vec{x}_1{}^*) = \\ &= \begin{pmatrix} A \\ (z_1{}^* + z_2{}^*)\sin\varphi \\ -(z_2{}^* - z_1{}^*)\cos\varphi \end{pmatrix} + \cos t \begin{pmatrix} H\cos\varphi - x_2{}^* + x_1{}^* \\ G - (y_2{}^* - y_1{}^*)\cos\varphi \\ -(y_1{}^* + y_2{}^*)\sin\varphi \end{pmatrix} + \\ &+ \sin t \begin{pmatrix} -G\cos\varphi + y_2{}^* - y_1{}^* \\ H - (x_2{}^* - x_1{}^*)\cos\varphi \\ -(x_1{}^* + x_2{}^*)\sin\varphi \end{pmatrix}. \end{aligned}$$

For further simplification we have put

$$(10) \qquad \begin{aligned} G &:= (-C + B\cos\varphi)/\sin^2\varphi \quad \text{and} \\ H &:= (D + A\cos\varphi)/\sin^2\varphi. \end{aligned}$$

Using $\vec{d}_0(t, \vec{x}_1{}^*, \vec{x}_2{}^*) = \vec{o}$ we determine $\vec{x}_1{}^*$. This gives the new representation of the relative motion $\Sigma_2{}^* \setminus \Sigma_1{}^*$. Formula (9) shows, that $\Sigma_2{}^* \setminus \Sigma_1{}^*$ may be gained by starting with the generating Darboux motions $\Sigma_1{}^* \setminus \Sigma_0$ and $\Sigma_2{}^* \setminus \Sigma_0$ parametrized by

$$\vec{x}_0(t, \vec{x}_1{}^*) := \begin{pmatrix} -A/2 \\ -z_1{}^* \sin\varphi \\ -z_1{}^* \cos\varphi \end{pmatrix} + \begin{pmatrix} -0.5H\cos\varphi - x_1{}^* \\ -G/2 - y_1{}^* \cos\varphi \\ y_1{}^* \sin\varphi \end{pmatrix} \cos t + \\ + \begin{pmatrix} 0.5G\cos\varphi + y_1{}^* \\ -H/2 - x_1{}^* \cos\varphi \\ x_1{}^* \sin\varphi \end{pmatrix} \sin t \qquad (t \in [0, 2\pi])$$

(11) and

$$\vec{x}_0(t, \vec{x}_2{}^*) := \begin{pmatrix} A/2 \\ z_2{}^* \sin\varphi \\ -z_2{}^* \cos\varphi \end{pmatrix} + \begin{pmatrix} 0.5H\cos\varphi - x_2{}^* \\ G/2 - y_2{}^* \cos\varphi \\ -y_2{}^* \sin\varphi \end{pmatrix} \cos t + \\ + \begin{pmatrix} -0.5G\cos\varphi + y_2{}^* \\ H/2 - x_2{}^* \cos\varphi \\ -x_2{}^* \sin\varphi \end{pmatrix} \sin t \qquad (t \in [0, 2\pi])$$

instead of (2).

If we now reflect Σ_0 and $\Sigma_2{}^*$ (11) with respect to the z_0- ($z_2{}^*$-)axis into $\widetilde{\Sigma}_0$, $\widetilde{\Sigma}_2$, resp., the motion $\Sigma_2{}^* \setminus \Sigma_0$ obtains the representation

$$\text{(12)} \quad \widetilde{\Sigma}_2/\widetilde{\Sigma}_0 \ldots \widetilde{\vec{x}_0}(t, \widetilde{\vec{x}_2}) = \begin{pmatrix} -A/2 \\ -\widetilde{z}_2 \sin\varphi \\ -\widetilde{z}_2 \cos\varphi \end{pmatrix} + \begin{pmatrix} -\widetilde{x}_2 - H/2\cos\varphi \\ -G/2 - \widetilde{y}_2 \cos\varphi \\ \widetilde{y}_2 \sin\varphi \end{pmatrix} \cos t + \\ + \begin{pmatrix} 0.5G\cos\varphi + \widetilde{y}_2 \\ -H/2 - \widetilde{x}_2 \cos\varphi \\ \widetilde{x}_2 \sin\varphi \end{pmatrix} \sin t.$$

This is exactly the representation (11) of the generating Darboux motion $\Sigma_1{}^* \setminus \Sigma_0$ if we put $x_0 = \widetilde{x}_0$, $y_0 = \widetilde{y}_0$, $z_0 = \widetilde{z}_0$ and $x_1 = \widetilde{x}_2$, $y_1 = \widetilde{y}_2$, $z_1 = \widetilde{z}_2$. Therefore we have: $\Sigma_1{}^* \setminus \Sigma_0$ may be gained by reflecting the generating Darboux motion $\Sigma_2{}^* \setminus \Sigma_0$ with respect to a straight line fixed in Σ_0. Thus our relative motion is line-symmetric in the sense of J. Krames [5]–[11]. $\Diamond$

The basic surface of this motion is generated as path surface Γ_1 of the z_0-axis under the inverse motion $\Sigma_0 \setminus \Sigma_1{}^*$. As $\Sigma_0 \setminus \Sigma_1{}^*$ is the inverse of a Darboux motion, this surface Γ_1 in the general case is algebraic of degree 4 — it first was used by J. Krames [10] in order to define special line-symmetric motions. For special cases (see our examples) the degree of Γ_1 may be lower.

4. Spherical paths under the relative motion. In following we use the representation (11) of our generating Darboux motions. We omit the "$*$" in the following. J. Krames [10] has shown, that at least a two-parametric family of points of Σ_2 under the relative motion $\Sigma_2 \setminus \Sigma_1$ is moved on spheres centered in Σ_1. As our concept allows a (new and quick) determination of these points we will give it here:

Corresponding points $\vec{x}_1$ and $\vec{x}_2$, fixed in the moving frames Σ_1 and Σ_2, resp., which in Σ_0 hold fixed distances during the two Darboux motions (we may say their distances are fixed under the relative motion $\Sigma_2 \setminus \Sigma_1$), have the following property: The vectors $\vec{d}_0(t, \vec{x}_1, \vec{x}_2) := = \vec{a} + \vec{b}\cos t + \vec{c}\sin t$ (9) with

$$\vec{a} := \begin{pmatrix} A \\ (z_1 + z_2)\sin\varphi \\ -(z_2 - z_1)\cos\varphi \end{pmatrix}, \qquad \vec{b} := \begin{pmatrix} H\cos\varphi - x_2 + x_1 \\ G - (y_2 - y_1)\cos\varphi \\ -(y_1 + y_2)\sin\varphi \end{pmatrix}, \tag{13}$$
$$\vec{c} := \begin{pmatrix} -G\cos\varphi + y_2 - y_1 \\ H - (x_2 - x_1)\cos\varphi \\ -(x_1 + x_2)\sin\varphi \end{pmatrix}$$

have constant length, iff the identity $\vec{d}_0(t, \vec{x}_1, \vec{x}_2).\frac{\partial}{\partial t}\vec{d}_0(t, \vec{x}_1, \vec{x}_2) = = 0$ holds for all $t \in [0, 2\pi]$. The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ only depend of the constants of the two motions and on the choice of the moving points $\vec{x}_1$, $\vec{x}_2$ resp. — they do not depend on the time parameter t. Therefore $\vec{d}_0(t, \vec{x}_1, \vec{x}_2) \cdot \frac{\partial}{\partial t}\vec{d}_0(t, \vec{x}_1, \vec{x}_2) = 0$ is equivalent to

$$\begin{aligned} 0 = (\vec{a}.\vec{c})\cos t + (\vec{a}.\vec{b})\sin t + (\vec{b}.\vec{c})\cos 2t - \\ - 0.5(\vec{c}^2 - \vec{b}^2)\sin 2t \quad \text{for all} \quad t \in [0, 2\pi]. \end{aligned} \tag{14}$$

As $\{\cos t, \sin t, \cos 2t, \sin 2t\}$ are linearly independent, the above equation yields 4 characteristic equations for the coordinates of corresponding points $\vec{x}_1$, $\vec{x}_2$ whose distance remains fixed under the relative motion $\Sigma_2 \setminus \Sigma_1$:

$$\begin{aligned} \vec{a}.\vec{b} &= 0 \qquad & \vec{a}.\vec{c} &= 0 \\ \vec{b}.\vec{c} &= 0 \qquad & \vec{b}^2 &= \vec{c}^2. \end{aligned} \tag{15}$$

Computation of these equations yields

$$\begin{aligned} &-Ax_2 - y_2 z_1 \sin 2\varphi + z_2(G\sin\varphi + y_1 \sin 2\varphi) = -A(H\cos\varphi + x_1) - Gz_1 \sin\varphi \\ &-x_2 z_1 \sin 2\varphi + Ay_2 + z_2(H\sin\varphi + x_1\sin 2\varphi) = A(G\cos\varphi + x_1) - Hy_1\sin\varphi \\ &\quad x_2 y_1 + y_2 x_1 = -HG/2 \\ &-x_2 x_1 + y_2 y_1 = (H^2 - G^2)/4. \end{aligned} \tag{16}$$

The correspondence between "linked points" of Σ_1 and Σ_2 in the first two coordinates reads

$$(17) \qquad x_2 = \frac{R^2(-x_1 \cos 2\alpha + y_1 \sin 2\alpha)}{x_1^2 + y_1^2}, \qquad y_2 = \frac{R^2(x_1 \sin 2\alpha + y_1 \cos 2\alpha)}{x_1^2 + y_1^2}$$

with abbreviations $H := 2R\cos\alpha$, $G := -2R\sin\alpha$ and certain reals R and α. Thus we have: If it is possible to link points of Σ_1 and Σ_2 by sticks with constant length, the ground projection of corresponding points of Σ_1 and Σ_2 *may be gained by an inversion with respect to a circle, followed by a displacement* (see also the results of R. Bricard [3] and J. Krames [10]).

The relation between the z-coordinates of corresponding points reads

$$(18) \qquad z_2 = \frac{A(2R\cos\alpha\cos\varphi + x_1 - x_2) + (-2R\sin\alpha\sin\varphi - y_2 \sin 2\varphi)z_1}{-2R\sin\alpha\sin\varphi + y_1 \sin 2\varphi},$$

where x_2 and y_2 have to be taken from (17). In (16) there remains an additional equation for z_1, which gives one constraint to the linked points in the first system. After some algebra we get

$$(19) \qquad \begin{aligned} &\big[(2y_1 + G\cos\varphi)^2 + (2x_1 + H\cos\varphi)^2 + (G^2 + H^2)\sin^2\varphi\big] * \\ &* \big[A(y_1 G + x_1 H + 2(x_1^2 + y_1^2)\cos\varphi) - z_1(Hy_1 - Gx_1)\sin 2\varphi)\big] = 0. \end{aligned}$$

Therefore the points of Σ_1, which may be linked with points of Σ_2 (via (17), (18)) are situated on an algebraic surface (19) of degree 4: But our surface splits into two parts: A nonreal rotational cylinder and a hyperboloid of one sheet containing the z_1-axis Φ_1. As the planes $z_1 =$ $=$ const give circular intersections, this hyperboloid is called *orthogonal* (see J. Krames [10]).

Change of indices (and signs) gives the corresponding points in Σ_2 on another degenerating algebraic surface of order 4: Its equation reads

$$(20) \qquad \begin{aligned} &\big[(2y_2 - G\cos\varphi)^2 + (2x_2 - H\cos\varphi)^2 + (G^2 + H^2)\sin^2\varphi\big] * \\ &* \big[A(y_2 G + x_2 H - 2(x_2^2 + y_2^2)\cos\varphi) - z_2(Hy_2 - Gx_2)\sin 2\varphi)\big] = 0. \end{aligned}$$

This surface splits into two parts, too and is congruent to (19). The hyperboloid of one sheet belonging to the points of Φ_1 will be called Φ_2. Similar results have been gained by J. Krames in [10]. We sum up in

Theorem 3. *Given two linked Darboux motions $\Sigma_1 \setminus \Sigma_0$ and $\Sigma_2 \setminus \Sigma_0$ in a fixed space Σ_0, both parametrized with respect to the angle of rotation. Then in the general case the relative motion $\Sigma_2 \setminus \Sigma_1$ moves a two-parametric family of points of Σ_2 on spheres centered in Σ_1. The corresponding points are situated on algebraic surfaces of order 4 in both systems.*

Remarks. 1) It may happen that the two linked surfaces do not contain a twoparametric family of real points. Our result is an algebraic one and therefore makes use of the complex embedding of the real space.

2) The straight lines g_2, h_2 (5) and (6) are situated on the hyperboloid Φ_2. The points of h_2 via (17), (18) correspond to the point at infinity on the z_1-axis. The point-paths under $\Sigma_2 \setminus \Sigma_1$ therefore are rational quartics and situated in planes (like discussed earlier). These planes have equations $z_1 = \text{const}$. An analogous result holds for the inverse motion.

3) The points on the circles of constant height $z_1 = \text{const}$ on Φ_1 via (17), (18) belong to points on hyperbolas on Φ_2 — they are situated in planes parallel to the z_2-axis. Points on generators of Φ_1 meeting the z_1-axis via (17), (18) correspond to points on generators of Φ_2 meeting the straight line $g_2 = z_2$-axis.

5. The special case $\sin 2\varphi = 0$. Here we have two possibilities:

A) $\varphi = 0$: Then (3) shows, that the relative motion $\Sigma_2 \setminus \Sigma_1$ in this case is a *translation with rational trajectories of degree* 4.

B) $\varphi = \pi/2$: Then the relative motion $\Sigma_2 \setminus \Sigma_1$ has a *fixed direction* and therefore is *cylindrical*. In (7) and (8) the transformation of the last coordinates shall be of the shape $z_1{}^* = C/2 - z_1$, $z_2{}^* = C/2 - z_2$. Then all calculations proceed till formulas (19), (20) with $\varphi = \pi/2$. In the general case the real part of surface Φ_2 having spherical paths under $\Sigma_2 \setminus \Sigma_1$ splits into a plane (parallel to the fixed direction) and the plane at infinity. In the very special case $C = D = 0$ we have $G = H = 0$ in (19) and (20) — hence in this special case all points of Σ_2 under $\Sigma_2 \setminus \Sigma_1$ move on spheres. This gives a rational type of the well-known *Bricard-motion* [3], which was seen to be line-symmetric by J. Krames in [6].

6. Examples. We study examples generated by two (congruent) *axial Darboux motions* (1) with constants $a_1{}^* = -A/2$, $a_2{}^* = A/2$, $b_1{}^* = b_2{}^* = c_1{}^* = c_2{}^* = 0$, $d_1{}^* = A/2$, $d_2{}^* = -A/2$, $e_1{}^* = e_2{}^* := \varepsilon$ and $f_1{}^* = f_2{}^* = 0$ with real constants A, ε. *The axes of the two motions shall be orthogonal*: $\varphi = \pi/4$. Following (2), (3) and (10) we get constants $A, B = 0$, $C = -\varepsilon\sqrt{2}$, $D = -0.5A\sqrt{2}$, $E = F = 0$ and $G = 2\varepsilon\sqrt{2}$, $H = 0$. The transformations (7) and (8) give new coordinates

$$\begin{aligned} x_2{}^* &= A/2 - x_2 & x_1{}^* &= -A/2 - x_1 \\ y_2{}^* &= \varepsilon - y_2 & y_1{}^* &= -\varepsilon - y_1 \\ z_2{}^* &= -\varepsilon - z_2 & z_1{}^* &= -\varepsilon - z_1. \end{aligned} \tag{21}$$

Omitting the "$*$" according to (19) the points of the algebraic surface

$$[(y_1+\varepsilon)^2+x_1^2+\varepsilon^2]*\\ *[A(2\varepsilon y_1+x_1^2+y_1^2)+2\varepsilon x_1 z_1]=0 \tag{22}$$

are moved on spheres under $\Sigma_1 \setminus \Sigma_2$. The corresponding points in the system Σ_2 belong to a surface congruent to (22). For corresponding points formula (17) now reads

$$x_2=\frac{2\varepsilon^2 x_1}{x_1^2+y_1^2}, \qquad y_2=\frac{-2\varepsilon^2 y_1}{x_1^2+y_1^2}. \tag{23}$$

As the constant A has no influence on (23) the ground-projection of our relationship may be studied in the case

A) $A = 0$, $\varepsilon \neq 0$: The axes of the generating axial Darboux motions meet at a point. Then the real part (second factor in (22)) splits into two orthogonal planes. Points of the plane $z_1 = 0$ may be linked with points of the plane $z_2 = 0$ via the inversion (23). From another starting point this situation was reached in the paper [14]. There two of the neighbour facets of a moveable model of a cube have been linked by 4 sticks of constant length. Fig. 1 shows this situation — it shows parts of the planes $z_1 = 0$ and $z_2 = 0$, the centers $I_{1,2} \in \Sigma_1$, $I_{2,1} \in \Sigma_2$ and circles $k_{1,2} \in \Sigma_1$, $k_{2,1} \in \Sigma_2$ of inversion, too. Here 6 connecting sticks have been drawn.

Figure 1.

B) $A \neq 0$, $\varepsilon = 0$: Here the generating Darboux-motions are pure *rotations*. Their axes do not meet. This is a very special situation of *Bennett's isogram*. The surface (22) splits into complex planes as J. Krames has shown in [9]. In our special case the basic surface Γ_1 of this line-symmetric motion is gained by rotating a straight line round a fixed axis — therefore Γ_1 is a rotational hyperboloid of one sheet.

Figure 2.

C) $A \neq 0$, $\varepsilon \neq 0$: This case gives a partial motion of the mechanism treated in [16]. Fig. 2 shows the situation: One hyperboloid and some possible links are shown in two different views. There the sticks connect points of two circles of the hyperboloid with corresponding points on hyperbolas situated on the second hyperboloid. This second hyperboloid is not shown here in order to give a more instructive picture.

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