Mathematica Pannonica
7/2 (1996), 191 – 196

SEPARATION BY MONOTONIC FUNCTIONS

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Received: August 1995

MSC 1991: 39 B 72, 26 A 51, 26 E 25

Keywords: Separation theorem, monotonic functions, quasiconvex and quasiconcave functions, selections.

Abstract: It is shown that real functions f and g defined on an arbitrary interval I can be separated by a monotonic function iff

$$f\big(tx+(1-t)y\big) \le \max\big\{g(x), g(y)\big\}$$

and

$$g\big(tx+(1-t)y\big) \ge \min\big\{f(x), f(y)\big\}$$

for all $x, y \in I$ and $t \in [0,1]$. Some results on the existence of monotonic selections of multifunctions and on the Hyers–Ulam stability of the monotonicity are also presented.

1. Introduction

The aim of this note is to characterize real functions f, g defined on an interval $I \subset \mathbb{R}$ which can be separated by a monotonic function.

This problem is connected with quasiconvex and quasiconcave functions and leads to functional inequalities

$$f\big(tx + (1-t)y\big) \leq \max\big\{g(x), g(y)\big\}$$

and

$$g\big(tx + (1-t)y\big) \geq \min\big\{f(x), f(y)\big\}.$$

The first of them appeared previously in a paper of J. Smolarz [3] and is equivalent to the fact that there exist a quasiconvex function $h : I \longrightarrow \mathbb{R}$ such that $f \leq h \leq g$. The results presented by us are related to a sandwich theorem obtained recently by K. Nikodem and Sz. Wąsowicz [2]. It states that there exists an affine function separating f and g iff

$$f\big(tx + (1-t)y\big) \leq tg(x) + (1-t)g(y)$$

and

$$g\big(tx + (1-t)y\big) \geq tf(x) + (1-t)f(y).$$

The first of the above inequalities implies that f and g can be separated by a convex function (cf. K. Baron, J. Matkowski and K. Nikodem [1]).

As an application of our separation theorem we obtain a result on the existence of monotonic selections of multifunctions whose values are compact intervals in $\mathbb{R}$. We also get a stability result of Hyers–Ulam type for monotonic functions.

Let us recall that a function $f : I \to \mathbb{R}$ is quasiconvex if

$$f\big(tx + (1-t)y\big) \leq \max\big\{f(x), f(y)\big\}, \quad x, y \in I,\ t \in [0,1];$$

it is quasiconcave if

$$f\big(tx + (1-t)y\big) \geq \min\big\{f(x), f(y)\big\}, \quad x, y \in I,\ t \in [0,1].$$

Obviously, a function $f : I \longrightarrow \mathbb{R}$ is monotonic iff it is quasiconvex and quasiconcave.

2. A separation theorem

Our main result reads as follows.

Theorem. *Let $I \subset \mathbb{R}$ be an arbitrary interval and $f,\ g : I \to \mathbb{R}$ be given functions. The following properties are equivalent:*

(a) *there is a monotonic function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$;*

(b) *there are functions h_1, $h_2 : I \to \mathbb{R}$, h_1–quasiconcave, h_2-quasiconvex, such that $f \leq h_1 \leq g$ and $f \leq h_2 \leq g$;*

(c) *there are functions h_1, $h_2 : I \to \mathbb{R}$, h_1–quasiconcave, h_2–quasiconvex, such that $f \leq h_1 \leq h_2 \leq g$;*

(d) *for all x, $y \in I$ and $t \in [0,1]$ the following inequalities hold*

$$
(1)\qquad \begin{aligned} f\big(tx+(1-t)y\big) &\le \max\big\{g(x),g(y)\big\}\\ g\big(tx+(1-t)y\big) &\ge \min\big\{f(x),f(y)\big\}. \end{aligned}
$$

Proof. Implication (a)$\Longrightarrow$(d) follows from the fact that every monotonic function is both quasiconvex and quasiconcave.

To prove that (d) implies (c) consider the functions h_1, $h_2 : I \to \to \mathbb{R}$ defined by

$$
(2)\qquad h_1(u) := \sup\big\{ \min\{f(x),f(y)\} : x \le u \le y,\, x,y \in I\big\}
$$

$$
(3)\qquad h_2(u) := \inf\big\{ \max\{g(x),g(y)\} : x \le u \le y,\, x,y \in I\big\}.
$$

By (1) the definitions are correct and

$$
(4)\qquad f(u) \le h_2(u) \quad \text{and} \quad h_1(u) \le g(u)
$$

for every $u \in I$. Moreover, $f \le h_1$ and $h_2 \le g$. We will show that $h_1 \le h_2$, h_1 is quasiconcave and h_2 is quasiconvex. Suppose, contrary to our claim, that there exist $w \in I$ such that $h_1(w) > h_2(w)$. Then there exist $x \le w \le y$ and $u \le w \le v$ such that

$$
(5)\qquad \min\big\{f(x),f(y)\big\} > \max\big\{g(u),g(v)\big\}.
$$

If $x \le u$, then from (2) and (5) it follows that

$$
h_1(u) \ge \min\big\{f(x),f(y)\big\} > g(u)
$$

which contradicts (4). If $u \le x$, then from (5) and (3) we get

$$
f(x) > \max\big\{g(u),g(v)\big\} \ge h_2(x),
$$

contrary to (4). These contradictions show that $h_1 \le h_2$.

Now we will prove that h_2 is quasiconvex (the quasiconcavity of h_1 follows similarly). Suppose that it is false. Then there exist $x \le u \le y$ such that

$$
h_2(u) > \max\big\{h_2(x),h_2(y)\big\}.
$$

By the definition of h_2 we can find $\alpha \le x \le \beta$ and $\gamma \le y \le \delta$ such that

$$
(6)\qquad h_2(u) > \max\big\{g(\alpha),g(\beta),g(\gamma),g(\delta)\big\}.
$$

However $\alpha \le u \le \delta$, which implies that

$$
h_2(u) \le \max\big\{g(\alpha),g(\delta)\big\}.
$$

This contradicts (6) and proves that h_2 is quasiconvex.

Implication (c)$\Longrightarrow$(b) is obvious.

To prove that (b) implies (a) assume first that $\sup\{f(z) : z \le \le x\} < \infty$ and $\inf\{g(z) : z \le x\} > -\infty$ for any $x \in I$. Define m_1, $m_2 : I \longrightarrow \mathbb{R}$ by

$$
m_1(x) := \sup\big\{f(z) : z \le x\big\}
$$

$$
m_2(x) := \inf\big\{g(z) : z \le x\big\}.
$$

It is evident that $f \le m_1$, $m_2 \le g$, m_1 is nondecreasing and m_2 is nonincreasing. We will show that at least one of these functions separates f and g. Assume the contrary. Then there are a, $b \in I$ such that $m_1(a) > g(a)$ and $m_2(b) < f(b)$. Without less of generality we may assume that $a \le b$ (otherwise we change f to $-f$ and g to $-g$, which interchanges the roles of h_1, h_2 and m_1, m_2). Let

$$0 < \varepsilon < \frac{1}{2} \min \left\{ m_1(a) - g(a), f(b) - m_2(b) \right\}. \tag{7}$$

By the definition of m_1 and m_2 there are x_1, $x_2 \in I$, $x_1 \le a$, $x_2 \le b$ such that

$$f(x_1) > m_1(a) - \varepsilon \quad \text{and} \quad g(x_2) < m_2(b) + \varepsilon. \tag{8}$$

If $x_1 \le x_2$, then x_2, $a \in [x_1, b]$ and by (b), (7) and (8) we get

$$h_1(x_2) \le g(x_2) < m_2(b) + \varepsilon < f(b) \le h_1(b)$$

and

$$h_1(a) \le g(a) < m_1(a) - \varepsilon < f(x_1) \le h_1(x_1),$$

which contradicts the quasiconcavity of h_1 on $[x_1, b]$. If $x_2 < x_1$, then $x_1 \in [x_2, a]$ and by (8), the definition of m_2 and (7) we obtain

$$h_2(x_2) \le g(x_2) < m_2(b) + \varepsilon \le g(a) + \varepsilon$$

$$h_2(a) \le g(a) < g(a) + \varepsilon$$

and

$$h_2(x_1) \ge f(x_1) > m_1(a) - \varepsilon > g(a) + \varepsilon,$$

which contradicts the quasiconvexity of h_2 on $[x_2, a]$.

Thus in any case we get a contradiction showing that at least one of the functions m_1, m_2 separates f and g.

Now we will deal with the existence of the functions m_1, m_2. As f is bounded from above by h_2 and therefore by $\max\left\{h_2(\alpha), h_2(\beta)\right\}$ on every compact interval $[\alpha, \beta] \subset I$ (and similarly g by h_1 from below), the only possibility for the nonexistence of m_1 (or m_2) is given if f (respectively g) is unbounded at the left border point of I. We need to consider the following three cases.

(i) Suppose that $\sup\left\{f(z) : z \le x\right\} = \infty$ and $\inf\left\{g(z) : z \le \le x\right\} = -\infty$ for some $x \in I$. Then, by the arguments given above, for any $y \le x$ we have $\sup\left\{f(z) : z \le y\right\} = \infty$ and $\inf\left\{g(z) : z \le y\right\} = = -\infty$. Thus there is a point $z_1 \le x$ such that $f(z_1) > 0$. Furthermore, we can find a point $z_2 < z_1$ such that $g(z_2) < 0$, and a point $z_3 < z_2$ such that $f(z_3) > 0$. Since $f \le h_1 \le g$, this is a contradiction to the quasiconcavity of h_1.

(ii) Suppose that $\sup\left\{f(z) : z \le x\right\} = \infty$ for some $x \in I$ and m_2 exists. Fix $x_0 \in I$ and choose $y_0 < x_0$ such that $f(y_0) > g(x_0)$. For any $y \in I$, $y \le y_0$, we may restrict m_2 to the interval $[y, \infty) \cap I$ and define m_1 as previously on this interval. Because $y_0 < x_0$ and $f(y_0) > g(x_0)$, the function m_1 can not separate f and g;, so, only the restriction of m_2 works as a separating function on $[y, \infty] \cap I$. Since this is valid for any $y \in I$, $y \leq y_0$, we conclude that m_2 separates f and g on I.

(iii) The case where m_1 exists but $\inf\{g(z) : z \leq x\} = -\infty$ for some $x \in I$ can be treated like (ii).

This finishes the proof. $\Diamond$

Remark. The above theorem can be proved in a different manner. For instance, since the implications (a)$\Longrightarrow$(c)$\Longrightarrow$(b)$\Longrightarrow$(d) are obvious and (d)$\Longrightarrow$(b) follows by the result of Smolarz [3], it is enough to show that (b)$\Longrightarrow$(a). The implication (c)$\Longrightarrow$(a) can be also obtained from the fact that quasiconvex and quasiconcave functions defined on an interval are either monotonic or unimodal (i.e. consist of two monotonic segments). Then it remains to prove that (d)$\Longrightarrow$(c), because the implications (a)$\Longrightarrow$ $\Longrightarrow$(b)$\Longrightarrow$(d) are evident. However, the proof presented by us is direct and it contains some ideas which may be interesting in themselves.

3. Applications

Let ε be a nonnegative constant. We say that a function $f : I \longrightarrow \mathbb{R}$ is ε–monotonic if

$$\min\{f(x), f(y)\} - \varepsilon \leq f(tx + (1-t)y) \leq \max\{f(x), f(y)\} + \varepsilon$$

for all x, $y \in I, t \in [0,1]$.

As an immediate consequence of our Theorem we obtain the following stability result of Hyers–Ulam type for monotonic functions.

Corollary 1. *A function $f : I \longrightarrow \mathbb{R}$ is ε-monotonic if and only if there exists a monotonic function $\varphi : I \longrightarrow \mathbb{R}$ such that*

$$| f(x) - \varphi(x) | \leq \frac{\varepsilon}{2}, \qquad x \in I. \tag{9}$$

Proof. If f is ε-monotonic, then the inequalities (1) hold with $g = f + \varepsilon$. By the Theorem there exists a monotonic function $h : I \longrightarrow \mathbb{R}$ such that $f \leq h \leq f + \varepsilon$. Putting $\varphi = h - \frac{\varepsilon}{2}$ we get a monotonic function satisfying (9). Now, assume that f satisfies (9) with a monotonic function φ. Then, for arbitrary x, $y \in I$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq \varphi(tx + (1-t)y) + \frac{\varepsilon}{2} \leq \max\{\varphi(x), \varphi(y)\} + \frac{\varepsilon}{2} \leq$$
$$\leq \max\{f(x), f(y)\} + \varepsilon.$$

Similarly

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\} - \varepsilon,$$

which ends the proof. $\Diamond$

Recall that a function $f : I \longrightarrow \mathbb{R}$ is the selection of a multifunction $\Phi : I \longrightarrow \mathbf{n}(\mathbb{R})$ (where $\mathbf{n}(\mathbb{R})$ denotes the family of all non-empty subsets of $\mathbb{R}$) if $f(x) in \Phi(x)$ $x \in I$. As a consequence of our Theorem we get also the following result on the existence of monotonic selections. Here $\mathbf{cc}(\mathbb{R})$ denotes the family of all compact intervals in $\mathbb{R}$ and $\mathrm{conv}(A)$ — the convex hull of a set A.

Corollary 2. *A multifunction* $\Phi : I \longrightarrow \mathbf{cc}(\mathbb{R})$ *has a monotonic selection if and only if*

$$\Phi(tx + (1-t)y) \cap \mathrm{conv}(\Phi(x) \cup \Phi(y)) \neq \emptyset \tag{10}$$

for all $x,\ y \in I,\ t \in [0,1]$.

Proof. Let us put $f(x) := \inf \Phi(x)$, and $g(x) := \sup \Phi(x)$ $x \in I$. Then $\Phi(x) = [f(x), g(x)]$ and

$$\mathrm{conv}(\Phi(x) \cup \Phi(y)) = [\min\{f(x), f(y)\}, \max\{g(x), g(y)\}].$$

Hence Φ satisfies (10) iff f and g satisfy (1), and a function $h : I \longrightarrow \mathbb{R}$ is a selection of Φ iff it separates f and g. So, to finish the proof it is enough to apply the Theorem. $\diamondsuit$

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