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MEASURE AND CATEGORY – SOME NON-ANALOGUES

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Abstract: Several results from the area of infinite series are established showing that some subsets of the interval $[0,1]$ for which certain series are convergent are of first category.

1. Introduction

"Measure and Category" is the title of a well–known and highly interesting book by J.C. Oxtoby; [8]. In it one finds numerous analogues and non-analogues between measure and category. A non-analogue that seems not widely known and does not appear in Oxtoby's book is the following. For each $x \in (0,1]$ let

$$x = \sum_{n=1}^{\infty} \frac{e_n(x)}{2^n}$$

be the non-terminating binary representation of x. The number x is said to be normal to base 2, in case the following limit exists:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e_k(x) = \frac{1}{2}.$$

It is known for quite some time that the Lebesgue measure of the set

$$N_2 = \{x \in [0,1] \big| x : \text{ normal to base } 2\}$$

is equal to 1. On the other hand, it seems less well known that N_2 is a set of first Baire category. Hence, N_2 is large in measure but small in category; in this case there is a non-analogue between measure and category.

Another result concerning series is the following: It is long known ([4]) that if

$$\sum_{n=1}^{\infty} a_n^2$$

converges then the random series

$$\sum_{n=1}^{\infty} \pm a_n$$

converges for almost all choices of the signs + and −. More precisely: If

$$x = \sum_{n=1}^{\infty} \frac{e_n(x)}{2^n}, \quad x \in (0,1],$$

is again the non-terminating binary expansion of x, then the set

$$M = \left\{ x \in (0,1] \Big| \sum_{n=1}^{\infty} (2e_n(x) - 1)a_n \text{ is convergent } \right\}$$

has measure one.

2. Results

Our first result shows that the set M just defined might be of the first Baire category since the following is true:

Theorem 1. *If*

$$\sum_{n=1}^{\infty} a_n$$

is not absolutely convergent, then the set

$$A = \left\{ x \in [0,1] \Big| \sum_{n=1}^{\infty} (2e_n(x) - 1)a_n \text{is convergent} \right\}$$

is of the first category.

Proof. The set A can be expressed in the following form:

$$A = \bigcap_{k=1}^{\infty} \left(\bigcup_{N=1}^{\infty} \left\{ x \in [0,1] \Big| \left| \sum_{n=i}^{j} (2e_n(x) - 1)a_n \right| < \frac{1}{k}, \quad \forall j > i \geq N \right\} \right) =$$

$$= \bigcap_{k=1}^{\infty} \left(\bigcup_{N=1}^{\infty} A_{N,k} \right),$$

the definition of the sets $A_{N,k}$ being obvious. It will be shown that each $A_{N,k}$ is nowhere dense. Take any

$$x = \sum_{n=1}^{m} \frac{x_n}{2^n} \quad \text{where} \quad m > N.$$

Since the series $\sum\limits_{n=1}^{\infty} a_n$ is not absolutely convergent, there exists an 'extension'

$$x' = x + \sum_{n=m+1}^{\infty} \frac{y_n}{2^n}$$

of x such that

$$\sum_{n=m+1}^{s} (2y_n - 1)a_n = \sum_{n=m+1}^{s} |a_n| \geq \frac{1}{k}$$

holds. Denoting by

$$B(s) = \big\{ z \in [0,1] \big| e_n(z) = x_n \quad \text{for} \quad n = 1, 2, \ldots, m \quad \text{and}$$
$$e_n(z) = y_n \quad \text{for} \quad n = m+1, m+2, \ldots, s \big\}$$

it follows that $B(s) \cap A_{N,k} = \emptyset$. Since the choice of x was arbitrary and since the set $B(s)$ is an interval, the sets $A_{N,k}$ are nowhere dense and hence A is a set of first category. $\Diamond$

If $S = \{S_k\}$ is a sequence and if the arbitrarily chosen $x \in (0,1]$ has the non-terminating binary representation

$$x = \sum_{n=1}^{\infty} \frac{e_n(x)}{2^n}$$

then the subsequence of S determined by x and denoted by $S(x)$ is defined to be $\{S_{k_j}\}_{j=1}^{\infty}$ where k_j is the place where the j-th one appears in the sequence $\{e_1(x), e_2(x), \dots\}$. Buck and Pollard [2], have proved in 1943 the following

Theorem A. *If $S = \{S_k\}$ is a $(C,1)$-summable sequence for which*

$$\sum_{k=1}^{\infty} \frac{S_{j_k}^2}{k^2} < \infty$$

holds, then the Lebesgue measure of the set

$$\{x \in [0,1] \big| S(x) \text{is } (C,1)\text{-summable to the } (C,1)\text{-limit of } S\}$$

is one.

Szüsz [10] later, in 1968, gave a simple probabilistic proof of this result using the strong law of large numbers of Kolomogorov. Recently, the current authors rediscovered the fact that the category analogue of the Th. A does not hold. Namely, if $S = \{S_k\}$ is a divergent sequence and C is a regular matrix summability method then the set

$$A = \{x \in (0,1] \big| S(x) \text{ is } C\text{-summable } \}$$

is of the first category. This result was first published in [6]. We are indebted to Professor J. Fridy for pointing out this reference to us. He also informed us about a paper by T. Keagy, [5], containing the following two results.

Theorem $\mathbf{K_1}$. *If A is a non-Schur matrix with convergent column and if $S = \{S_k\}$ is a divergent sequence, then the set $\{x \in [0,1] \big| S(x)$ is A-summable$\}$ is of the first category.*

Remark. Since the class of non-Schur matrices with convergent columns strictly contains the class of regular matrices, Th. K_1 is a generalization of the theorem mentioned above that appears in [6].

Theorem $\mathbf{K_2}$. *Let A be a non-Schur matrix with convergent columns and let $S = \{S_k\}$ denote a divergent sequence. Then the set of A-summable rearrangements of S is of the first category.*

Remark. Here a word about terminology seems in order. Let $\mathfrak{P}$ denote the collection of all permutations of $\mathbb{N}$, the set of natural numbers, i.e. the collection of all injective mappings P of $\mathbb{N}$ onto itself. On $\mathfrak{P}$ a metric d is defined in the following way:

$$\text{if} \quad P_1, P_2 \in \mathfrak{P}, \quad \text{then} \quad \mathrm{d}(P_1,P_2) = 0 \quad \text{in case } P_1 = P_2$$

$$\text{whilst} \quad \mathrm{d}(P_1,P_2) = \frac{1}{j} \quad \begin{array}{l} \text{if } P_1(i) = P_2(i) \\ \text{if } i = 1,2,\dots,j-1 \\ \text{and } P_1(j) \neq P_2(j). \end{array}$$

In the metric space $(\mathfrak{P}, \mathrm{d})$ $\mathfrak{P}$ is then of the second category in itself; [3].

Our next results — Th. 2, 2′ and 3 — are extensions of Th. K_1. Every sequence $S = \{S_k\}$ is summed by some non-Schur matrix with convergent columns; then the following holds:

$$\{x \in (0,1] \big| S(x) \text{ is } A\text{-summable for some non-Schur matrix with convergent columns}\}$$

equals the interval $(0,1]$. Here it will be shown that if $S = \{S_k\}$ is a divergent sequence, then there exists a collection $\mathfrak{A}$ of non-Schur matrices with convergent columns, having cardinality of the continuum, such that the set

$$\{x \in (0,1] \big| S(x) \text{ is } A\text{-summable for some } A \in \mathfrak{A}\}$$

is of first category.

Of course, this result would be trivial in case all of the matrices in $\mathfrak{A}$ had the same convergence field. In connection with this remark the following example is informative.

Example 1. Let $\varepsilon > 0$. For $t \in [0, \varepsilon)$ define $A_t = a_{nk}(t)$ as follows:

$$a_{nk}(t) = \begin{cases} 1 + (-1)^{n+1}(t - \varepsilon) & \text{for} \quad k = 1, 2, \ldots, n, \\ \dfrac{(-1)^n t}{m_1(n)} & \text{for} \quad k \in B_1(n), \\ \dfrac{(-1)^{n+1}\varepsilon}{m_2(n)} & \text{for} \quad k \in B_2(n), \\ 0 & \text{otherwise}, \end{cases}$$

where $B_1(n)$ and $B_2(n)$ are for each n blocks of consecutive integers satisfying:

a) $B_1(n)$ lies to the 'left' of $B_2(n)$;
b) n is less than each integer in $B_1(n)$;
c) all of the blocks $\{B_1(n) \big| n \in \mathbb{N}\}$ and $\{B_2(n) \big| n \in \mathbb{N}\}$ are pairwise disjoint;
d) $m_i(n)$ is the number of elements of $B_i(n)$, $i = 1, 2$;
e) the set $\bigcup_{n=1}^{\infty} (B_1(n) \cup B_2(n))$ has natural density 0 in $\mathbb{N}$;
f) $\lim_{n \to \infty} m_i(n) = \infty$ for $i = 1, 2$.

A_t is a regular matrix summability method for each $t \in [0, \varepsilon)$. For each such t let x_t denote a sequence of 0's and 1's satisfying

$$\frac{1}{m_1(n)} \sum_{k \in B_1(n)} x_t(k) \longrightarrow \varepsilon \quad \text{as} \quad n \to \infty, \quad \text{and}$$

$$\frac{1}{m_2(n)} \sum_{k \in B_2(n)} x_t(k) \longrightarrow t \quad \text{as} \quad n \to \infty.$$

From this it is easy to derive that:

$$x_t \text{ is } A_t\text{-summable to 0 for } t \in [0, \varepsilon),$$

but x_t is not A_s-summable if $0 \leq t,\ s < \varepsilon$ and $t \neq s$ hold.

In the sequel the following notations will be used:

$$K(A, \varepsilon) := \{B \mid B \text{ is a non-Schur matrix with convergent columns and } ||A - B|| < \varepsilon\}$$

where

$$||C|| = \sup_p \left(\sum_{q=1}^{\infty} |c_{pq}| \right) \quad \text{if} \quad C = (c_{pq}).$$

Now our extensions of Th. K_1 will be presented.

Theorem 2. *If A is a non-Schur matrix with convergent columns and if $S = \{S_k\}$ is a bounded divergent sequence, then the set*

$$\{x \in (0, 1] \mid S(x) \text{is } B\text{-summable for some } B \in K(A, \varepsilon)\}$$

is of the first category for some $\varepsilon > 0$.

Proof. This proof follows the lines of the proof of Th. 3 in [5] and is divided into the consideration of two cases. Let $A = (A_{pq})$ and let $T(A, \varepsilon)$ denote the set

$$\{x \in (0, 1) \mid S(x) \text{ is } B\text{-summable for some } B \in K(A, \varepsilon)\}.$$

Further, J shall denote the collection of all subsequences of S.

CASE I. Suppose row p of A is not absolutely convergent. If $y \in J$ and $y = \{y_q\}$ is B-summable for some $B \in K(A, \varepsilon)$, $B = (B_{pq})$, then there exists an N such that

$$\left| \sum_{q=i}^{j} b_{pq} y_q \right| \leq \frac{1}{2} \quad \text{for every} \quad j > i \geq N.$$

From this follows that if $\varepsilon > 0$ is sufficiently small, then

$$\left| \sum_{q=i}^{j} a_{pq} y_q \right| \leq 1 \quad \text{for every} \quad j > i \geq N.$$

Next, the following set is defined:

$$E_N = \Big\{x \in (0,1] \Big| \text{ there exist } j > i \geq \mathbf{N} \text{ such that } \Big| \sum_{q=i}^{j} a_{pq}(S(x))_q \Big| > 1\Big\}.$$

By the above

$$T(A,\varepsilon) \subseteq \bigcup_{N=1}^{\infty} E_N^c.$$

Moreover, in [5] it is shown that the set on the right side is of the first category, so that the same must hold for $T(A,\varepsilon)$.

CASE II. Suppose each row of A to be absolutely convergent. Now Maddox (Lemma 1 in [5]) showed that if the matrix A is Schur there exists a divergent sequence x such that each subsequence of x is A-summable. Hence there is an $y \in J$ such that y is not A-summable. Therefore, there exists a $\delta > 0$ such that to every positive integer N integers $j_N > i_N \geq N$ exist so that

$$(*) \qquad \left| \sum_{q=1}^{\infty} a_{i_N q} y_q - \sum_{q=1}^{\infty} a_{j_N q} y_q \right| > \delta$$

holds. Next, let be

$$E_N = \Bigg\{ x \in (0,1] \Big| \text{ there exists } j > i \geq N \text{ such that}$$
$$\left| \sum_{q=1}^{\infty} a_{iq}(S(x))_q - \sum_{q=1}^{\infty} a_{jq}(S(x))_q \right| > \frac{\delta}{2} \Bigg\}.$$

Since each row of A is absolutely convergent and since S is bounded it follows that each $x \in E_N$ is contained in an interval that is a subset of E_N. By $(*)$ and the fact that the columns of A converge follows that each E_N is dense in $(0,1)$.

Now suppose $w \in J$, w being B-summable, $B = (b_{pg})$ and $B \in \in K(A,\varepsilon)$. Then there exists an N such that

$$\left| \sum_{q=1}^{\infty} b_{iq} w_q - \sum_{q=1}^{\infty} b_{jq} w_q \right| \leq \frac{\delta}{4}$$

is true for all $j > i \geq N$. This implies that

$$\left| \sum_{q=1}^{\infty} a_{iq} w_q - \sum_{q=1}^{\infty} a_{jq} w_q \right| \leq \frac{\delta}{2}$$

holds for every $j > i \geq N$, if only ε is sufficiently small. Thus it has been shown that

$$T(A,\varepsilon) \subseteq \bigcup_{N=1}^{\infty} E_N^c$$

and hence $T(A,\varepsilon)$ is of the first category. $\Diamond$

The following example shows that the condition of S being bounded in Th. 2 is necessary.

Example 2. Let $S_n = n$, $n = 1, 2, \dots$ and suppose that $y = \{y_q\}$ is any subsequence of $S = \{S_n\}$. By C denote the $(C,1)$-matrix. It will be shown that for each $\varepsilon > 0$ there exists a regular matrix method $C_\varepsilon \in K(C,\varepsilon)$ so that C_ε sums y to zero. For $n > 2$ the matrix method $C(n)$ is defined as follows: the first $n-1$ rows of $C(n)$ are the same as those of C. Then there exists a $q_n > n$ so that

$$\frac{1}{n}\sum_{q=1}^{n} y_q - \frac{1}{n} y_{q_n} < 0.$$

Therefore there exists a λ_n, $0 < \lambda_n < 1$, such that

$$\frac{1}{n}\sum_{q=1}^{n} y_q - \frac{\lambda_n}{n} y_{q_n} = 0.$$

The terms in the n-th row of $C(n)$ are to be $\frac{1}{n}$ in the first n places, $-\frac{\lambda_n}{n}$ in the q_n-th place and 0 in all other places. The remaining rows are defined similarly yielding with $C(n) = (c_{pq}^{(n)})$

$$\sum_{q=1}^{\infty} c_{pq}^{(n)} y_q = 0 \quad \text{for all} \quad p \geq n.$$

Obviously, each $C(n)$ is regular and $||C - C(n)|| \to 0$ as $n \to \infty$.

Despite this example, for unbounded sequences we have the following result:

Theorem 2′. *If A is a non-Schur matrix with convergent columns and if $S = \{S_k\}$ is an unbounded sequence, then for every $\varepsilon > 0$ there exists a collection $\mathfrak{A}$, having cardinality of the continuum, $\mathfrak{A} \subset K(A,\varepsilon)$, such that the set*

$$\big\{x \in (0,1] \big| S(x) \text{is } B\text{-summable for some } B \in \mathfrak{A}\big\}$$

is of the first category.

Proof. Let $\varepsilon > 0$ and let p be any positive integer. There exists an $A' = (a'_{pq})$, $A' \in K(A, \frac{\varepsilon}{2})$ such that row p of A' is not eventually zero. There also exists a collection $\mathfrak{A} \in K(A,\varepsilon)$, $\mathfrak{A}$ having cardinality of the continuum, such that if $B \in \mathfrak{A}$ and $a'_{pn} \neq 0$, then $|b_{pn}| > \frac{1}{2}|a'_{pn}|$. Let next be:

$$E_N = \{x \in (0,1] \,|\, \text{there exists an } n \geq N \text{ such that } |a'_{pn}(S(x))_n| > 1\}.$$

Clearly, each E_N is dense in $(0,1]$ and if $x \in E_N$ then an interval containing x is contained in E_N. The set

$$\{x \in (0,1] \,|\, S(x) \text{is } B\text{-summable for some } B \in \mathfrak{A}\}$$

is obviously a subset of $\bigcup_{N=1}^{\infty} E_N^c$ and is therefore of the first category. $\Diamond$

We now turn our attention to theorems dealing with rearrangements of sequences. If $S = \{S_n\}$ and $P \in \mathfrak{P}$, let $S(P)$ denote the rearrangement of S by P, i.e. $(S(P))_k = S_{P(k)}$ for each k.

Theorem 3. *If A is a non-Schur matrix with convergent columns and if $S = \{S_k\}$ is a bounded divergent sequence, then the set*

$$U(A,\varepsilon) = \{P \in \mathfrak{P} \,|\, S(P) \text{ is } B\text{-summable for some } B \in K(A,\varepsilon)\}$$

is of the first category in $(\mathfrak{P}, d)$ for some $\varepsilon > 0$.

Proof. CASE I. Suppose row p of A is not absolutely convergent. If $y = (y_q) \in H$, where H is the collection of all rearrangements of S, y being B-summable for some $B \in K(A,\varepsilon)$, $B = (b_{pq})$, then there exists an N such that

$$\left|\sum_{q=i}^{j} b_{pq} y_q\right| \leq \frac{1}{2} \quad \text{for every} \quad j > i \geq N.$$

From this follows that, for $\varepsilon > 0$ sufficiently small, the inequalities

$$\left|\sum_{q=i}^{j} b_{pq} y_q\right| \leq 1 \quad \text{for all} \quad j > i \geq N \quad \text{hold.}$$

Now let

$$E_N = \Big\{P \in \mathfrak{P} \,\Big|\, \text{there exist } j > i \geq N \text{ such that } \Big|\sum_{q=i}^{j} a_{pq}(S(P))_q\Big| > 1\Big\}.$$

Notice that if $P \in E_N$, $P' \in \mathfrak{P}$ and $P(q) = P'(q)$ for all $q = 1, 2, \ldots, j$, then $P' \in E_N$. Hence, by the definition of the metric d on $\mathfrak{P}$, E is an open set. Furthermore, by the argument used by Keagy in proving Th. 4 in [5], it follows that each E_N is dense in $\mathfrak{P}$. Therefore it has been shown that $U(A,\varepsilon) \subseteq \bigcup_{N=1}^{\infty} E_N^c$ holds for sufficiently small ε and that $\bigcup_{N=1}^{\infty} E_N^c$ and also $U(A,\varepsilon)$ is of the first category.

CASE II. Suppose each row of A is absolutely convergent. In this case the proof then follows exactly that in Case II in Th. 2. In place of the lemma of Maddox in [5], the following result of Keagy, [5], is used: If A is a non-Schur matrix and if $S = \{S_k\}$ is a divergent sequence then there exists a $P \in \mathfrak{P}$ such that $S(P)$ is not A-summable.

Example 2 shows also that the requirement that S is bounded in Th. 3 is necessary. However, we again do have the following

Theorem 3′. *If A is a non-Schur matrix with convergent columns and if $S = \{S_k\}$ is an unbounded sequence, then for every $\varepsilon > 0$ there exists a collection $\mathfrak{A}$ of cardinality of the continuum, $\mathfrak{A} \subset K(K, \varepsilon)$ such that the set*

$$\{P \in \mathfrak{P} \big| S(P) \text{is } B\text{-summable for some } B \in \mathfrak{A}\}$$

is of the first category.

The **proof** of this result is the same as that of the proof of Th. 2, only with x replaced by P and $(0, 1]$ replaced by $\mathfrak{P}$. ◊

Two **final remarks 1.** The rearrangement analogue of Th. 1 was proved by Agnew in [1]. **2.** A recent result of the type of results presented here is by Miller and can found in [7].

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