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UNIFORM CONVERGENCE OF LIENHARD'S SPLINE APPROXIMATIONS OF A GIVEN CONTINUOUS FUNCTION

Dedicated to o. Univ. Prof. Dr. Hans Vogler at the occasion of his 60th birthday

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Abstract: The paper is related to the article [1]. It is proved that a sequence of $L_{1,0}$- spline approximations of a given continuous function f in the interval $< a, b >$ converges in this interval uniformly to f (cf. [2]).

1. A sequence of $L_{1,0}$-spline approximations of a given function f in the interval $< a, b >$

In the interval $< a, b >$ of finite length $L = b - a > 0$ let us consider a (real) function $x_2 = f(x_1)$. Let $n \geq 2$ be a positive integer. We divide the interval $< a, b >$ into n equal intervals by the dividing points $a = x_1^{(1)} < x_1^{(2)} < \cdots < x_1^{(n)} < x_1^{(n+1)} = b$. We have $x_1^{(i)} = a + (i-1)h$, $i = 1, 2, \dots, n+1$, where $h = L/n$. We denote $f(x_1^{(i)}) = x_2^{(i)}$ and, further, $P_i = (x_1^{(i)}, x_2^{(i)})$, $i = 1, 2, \dots, n+1$. For every $n \geq 2$ we have $x_2^{(1)} = f(a)$, $x_2^{(n+1)} = f(b)$.

Through the points $P_1, P_2, \dots, P_n, P_{n+1}$ we fit an unclosed interpolation $L_{1,0}$-spline (see [1], Section 6), whose i-th arc P_iP_{i+1}, $i = 1, 2 \dots, n$, is parametrized with the aid of the polynomials

$$x_j = P_{x_j}^{(i)}(t) = (1, t, t^2, t^3) \circ A_{ij}^T \quad (j = 1, 2), \tag{1.1}$$

where

$$A_{ij}^T = C \circ \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}, b_j^{(i)}, b_j^{(i+1)}\right)^T, \tag{1.2}$$

$$C = \frac{1}{16} \begin{bmatrix} -1 & 9 & 9 & -1 & 4m_1 & -4m_1 \\ 1 & -11 & 11 & -1 & -4m_1 & -4m_1 \\ 1 & -1 & -1 & 1 & -4m_1 & 4m_1 \\ -1 & 3 & -3 & 1 & 4m_1 & 4m_1 \end{bmatrix} \tag{1.3}$$

(see [1], formula (5.2) for $Q = 1$, $p = 0$), where the parameter t varies in the interval $< -1, 1 >$. For $j = 1$ and $i = 1$, or $i = n$, we choose $x_1^{(0)} = a - h$, or $x_1^{(n+2)} = b + h$. Further, it is possible to choose the values $x_2^{(0)}$ and $x_2^{(n+2)}$ more or less arbitrarily, and we put $P_0 = (x_1^{(0)}, x_2^{(0)})$, $P_{n+2} = (x_1^{(n+2)}, x_2^{(n+2)})$.

In the matrix (1.3) m_1 is a real number different from zero. Further, $b_j^{(1)}, b_j^{(2)}, \dots \dots, b_j^{(n+2)}$ are vectors from the vector space V^2 whose components satisfy, for $j = 1, 2$, the following system of linear equations:

$$\begin{aligned} 16m_1 b_j^{(1)} + c_j b_j^{(2)} &= z_j, \\ 4m_1 b_j^{(1)} + 16m_1 b_j^{(2)} + 4m_1 b_j^{(3)} &= p_j^{(1)}, \\ 4m_1 b_j^{(2)} + 16m_1 b_j^{(3)} + 4m_1 b_j^{(4)} &= p_j^{(2)}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ 4m_1 b_j^{(n)} + 16m_1 b_j^{(n+1)} + 4m_1 b_j^{(n+2)} &= p_j^{(n)}, \\ d_j b_j^{(n+1)} + 16m_1 b_j^{(n+2)} &= u_j. \end{aligned} \tag{1.4}$$

Here, the real numbers c_j, d_j satisfy the inequalities $|c_j| < 16|m_1|$, $|d_j| < 16|m_1|$. Further, z_j, u_j are arbitrary real numbers, and for $k =$

$= 1, 2, \ldots, n$ we have

(1.5) $p_j^{(k)} = (P_{k-1} - P_k) - (P_k - P_{k+1}) - (P_{k+1} - P_{k+2}) + (P_{k+2} - P_{k+3})$

(see [1], Section 3). For $k = n$ we put $P_{n+3} = (x_1^{(n+3)}, x_2^{(n+3)})$, where $x_1^{(n+3)} = b + 2h$; further, it is possible to choose the value $x_2^{(n+3)}$ more or less arbitrarily.

By (1.5) we easily verify that for $j = 1$ we have $p_1^{(k)} = 0$, $k = 1, 2 \ldots, n$. Further, we put $z_1 = 0$, $u_1 = 0$ in the system (1.4) [for $j = 1$]. Since the corresponding matrix of the system has a dominant main diagonal due to the mentioned constraints concerning the numbers c_1, d_1, i.e. the matrix is regular, the system possesses only the trivial solution: $b_1^{(k)} = 0$, $k = 1, 2, \ldots, n + 2$. By (1.1), (1.2) [for $j = 1$] we then have

$$x_1 = P_{x_1}^{(i)}(t) = (1, t, t^2, t^3) \circ C \circ \begin{bmatrix} a + (i-2)h \\ a + (i-1)h \\ a + ih \\ a + (i+1)h \\ 0 \\ 0 \end{bmatrix} = \tag{1.6}$$

$$= a + ih - \frac{h}{2} + \frac{h}{2} t,$$

where $P_{x_1}^{(i)}$ is a function with definition domain $< -1, 1 >$ and range $< a + (i-1)h, a + ih >$. For the inverse function $[P_{x_1}^{(i)}]^{-1} :< a + (i - 1)h, a + ih > \rightarrow < -1, 1 >$ we then have

$$t = \left[P_{x_1}^{(i)}\right]^{-1}(x_1) = \frac{2}{h}(x_1 - a) - (2i - 1). \tag{1.7}$$

Upon substitution of (1.7) into (1.1) [for $j = 1$] we then obtain

$$x_2 = P_{x_2}^{(i)} \circ [P_{x_1}^{(i)}]^{-1}(x_1) = r_n^{(i)}(x_1) = \Bigg(1, \frac{2}{h}(x_1 - a) - \tag{1.8}$$

$$-(2i-1), \left[\frac{2}{h}(x_1 - a) - (2i-1)\right]^2, \left[\frac{2}{h}(x_1 - a) - (2i-1)\right]^3\Bigg) \circ$$

$$\circ C \circ \left(x_2^{(i-1)}, x_2^{(i)}, x_2^{(i+1)}, x_2^{(i+2)}, b_2^{(i)}, b_2^{(i+1)}\right)^T,$$

where $r_n^{(i)} = P_{x_2}^{(i)} \circ [P_{x_1}^{(i)}]^{-1}$ is a function with domain $< a + (i-1)h, a + ih >$.

For the chosen number n a given $x_1 \in< a,b >$ we determine the number $i = [(x_1 - a)/h] + 1$, where the square bracket denotes the integer part of the respective real number. If x_1 runs through the interval $< a,b)$, the i assumes the values $i = 1,2,\dots,n$. We have $i-1 = [(x_1 - -a)/h] \le (x_1-a)/h < [(x_1-a)/h]+1 = i$, i.e. $a+(i-1)h \le x_1 < a+ih$. We put

$$\begin{aligned}\frac{2}{h}(x_1-a)-(2i-1) &= 2\frac{x_1-a}{h} - 2\Big[\frac{x_1-a}{h}\Big] - 1 = \\ &= 2\Big\{\frac{x_1-a}{h} - \Big[\frac{x_1-a}{h}\Big]\Big\} \overset{\text{def}}{=} \Big\langle \frac{x_1-a}{h}\Big\rangle. \end{aligned} \tag{1.9}$$

In the interval $< a+(i-1)h, a+ih)$ it is then possible to represent (1.8) in the form

$$\begin{aligned} x_2 = r_n^{(i)}(x_1) &= \\ = \left(1, \Big\langle\frac{x_1-a}{h}\Big\rangle, \Big\langle\frac{x_1-a}{h}\Big\rangle^2, \Big\langle\frac{x_1-a}{h}\Big\rangle^3\right) &\circ \\ \circ C \circ \left(x_2^{(i-1)}, x_2^{(i)}, x_2^{(i+1)}, x_2^{(i+2)}, b_2^{(i)}, b_2^{(i+1)}\right)^T&. \end{aligned} \tag{1.10}$$

Hence, for $i = n$ this yields

$$r_n^{(n)}(b) = \lim_{x_1 \to b-} r_n^{(n)}(x_1) = \tag{1.11}$$

$$= (1,1,1,1) \circ C \circ \left(x_2^{(n-1)}, x_2^{(n)}, x_2^{(n+1)}, x_2^{(n+2)}, b_2^{(n)}, b_2^{(n+1)}\right)^T =$$

$$= \frac{1}{16}(0,0,16,0,0,0) \circ \left(x_2^{(n-1)}, x_2^{(n)}, x_2^{(n+1)}, x_2^{(n+2)}, b_2^{(n)}, b_2^{(n+1)}\right)^T =$$

$$= x_2^{(n+1)} = f(b).$$

By the symbol r_n we denote a function $r_n :< a,b > \to R^1$ with the following properties:

$$\begin{aligned} r_n|_{<a+(i-1)h, a+ih)} = r_n^{(i)} \text{ for } r_n \text{ for } i = 1,2,...,n, \\ r_n(b) = f(b). \end{aligned} \tag{1.12}$$

By the symbol $r_n|_I$ we denote the restriction of the function r_n to the interval $I = =< a+(i-1)h, a+ih)$. By (1.11) we have $r_n(b) = r_n^{(n)}(b)$.

2. Estimate of the norm of the inverse matrix of system (1.4)

Under the assumption that $|c_2| < 16|m_1|$, $|d_2| < 16|m_1|$ the matrix $A = (a_{kh})$ of the system (1.4) [for $j = 2$] has a dominant main diagonal, i.e.

$$\tag{2.1} \begin{aligned} &\min_k \Big\{ |a_{kk}| - \sum_{h \neq k} |a_{kh}| \Big\} = \\ &= \min\{8|m_1|, 16|m_1| - |c_2|, 16|m_1| - |d_2|\} = q > 0. \end{aligned}$$

For the operator norm of the respective inverse matrix A^{-1}, i.e. for a norm induced by the first norm of a vector, the inequality $\|A^{-1}\| \leq q^{-1}$ holds.

This can be easily proved. Let us put $b = (b_2^{(1)}, b_2^{(2)}, \dots, b_2^{(n+2)})^T$, $p = (z_2, p_2^{(1)}, p_2^{(2)}, \dots, p_2^{(n)}, u_2)^T$. Then the matrix representation of this system is $A \circ b = p$, consequently $b = A^{-1} \circ p$. Let

$$\|b\| = \max_k \left| b_2^{(k)} \right| = \left| b_2^{(m)} \right|, 1 \leq m \leq n+2.$$

Then

$$\|p\| = \|A \circ b\| = \max_k \Big| \sum_{h=1}^{n+2} a_{kh} b_2^{(h)} \Big| \geq \Big| \sum_{h=1}^{n+2} a_{mh} b_2^{(h)} \Big| =$$

$$= \Big| a_{mm} b_2^{(m)} + \sum_{h \neq m} a_{mh} b_2^{(h)} \Big| \geq |b_2^{(m)}| |a_{mm}| - \Big| \sum_{h \neq m} a_{mh} b_2^{(h)} \Big| \geq$$

$$\geq \left| b_2^{(m)} \right| \left| a_{mm} \right| - \left| b_2^{(m)} \right| \sum_{h \neq m} \left| a_{mh} \right| \geq \|b\| \cdot \min_k \Big\{ |a_{kk}| - \sum_{h \neq k} |a_{kh}| \Big\} = \|b\| q.$$

Hence it already follows that

$$\tag{2.2} \|A^{-1}\| = \sup_{p \neq 0} \frac{\|A^{-1} \circ p\|}{\|p\|} = \sup_{A \circ b \neq 0} \frac{\|b\|}{\|A \circ b\|} \leq q^{-1}.$$

3. Uniform convergence of $L_{1,0}$-spline approximations of a given continuous function f in the interval $< a, b >$

Let f be a continuous function in the interval $< a, b >$ of finite length $L = b - a > 0$. Then it is uniformly continuous in this interval. Thus, to a given $\varepsilon > 0$ there exists $\delta > 0$ such that for all points $x_1', x_1'' \in < a, b >$ whose distance $|x_1' - x_1''|$ is less than δ we have

$$|f(x_1') - f(x_1'')| < \frac{2\varepsilon}{9}. \tag{3.1}$$

We put

$$n_0 = \max\left\{1, \left[\frac{3L}{\delta}\right] + 1\right\}, \tag{3.2}$$

where the square bracket denotes the integer part of the respective real number. We divide the interval $< a, b >$ into $n > n_0$ equal intervals of length $h = L/n$. By Section 1 it is possible to choose the second coordinates of the points $P_0 = (a - h, x_2^{(0)})$, $P_{n+2} == (b + h, x_2^{(n+2)})$, $P_{n+3} = (b + 2h, x_2^{(n+3)})$ more or less arbitrarily. Thus, we shall assume that

$$\begin{gathered} \left|x_2^{(0)} - f(a)\right| < \frac{2\varepsilon}{9}, \left|x_2^{(n+2)} - f(b)\right| < \frac{2\varepsilon}{9}, \\ \left|x_2^{(n+2)} - x_2^{(n+3)}\right| < \frac{2\varepsilon}{9} \end{gathered} \tag{3.3}$$

holds. Further, by (3.1), (3.2), we have [see (1.5)]

$$\begin{aligned} \|p\| = \max\left\{|z_2|, |p_2^{(1)}|, |p_2^{(2)}|, \ldots, |p_2^{(n)}|, |u_2|\right\} \leq \\ \leq \max\{|z_2|, |u_2|, 8\varepsilon/9\}. \end{aligned} \tag{3.4}$$

By (2.1), (2.2), (3.4), we then have

$$\begin{aligned} |b_2^{(k)}| \leq \|b\| = \|A^{-1} \circ p\| \leq \|A^{-1}\|\|p\| \leq q^{-1}\|p\| \leq \\ \leq \frac{\max\{|z_2|, |u_2|, 8\varepsilon/9\}}{\min\{8|m_1|, 16|m_1| - |c_2|, 16|m_1| - |d_2|\}} \end{aligned} \tag{3.5}$$

for $k = 1, 2, \ldots, n+2$. In what follows we shall assume that $|z_2| \leq 8\varepsilon/9$, $|u_2| \leq 8\varepsilon/9$, $|c_2| \leq 8|m_1|$, $|d_2| \leq 8|m_1|$. Then we have $\max\{|z_2|, |u_2|, 8\varepsilon/ /9\} = 8\varepsilon/9$, $\min\{8|m_1|, 16|m_1| - |c_2|, 16|m_1| - |d_2|\} = 8|m_1|$, and thus we have, by (3.5),

$$|b_2^{(k)}| \leq \frac{\varepsilon}{9|m_1|} \tag{3.6}$$

for $k = 1, 2, \ldots, n + 2$.

For $n > n_0$ [see (3.2)] and $z \in < a, b)$, we have for the respective $i = [(z-a)/h] + +1 : |x_1^{(k-1)} - z| < \delta$ for $k = i, i+1, i+2, i+3$. By (3.1), (3.3), we then have $|x_2^{(k-1)} - f(z)| < < 2\varepsilon/9$ for $k = i, i + 1, i + 2, i + 3$. For these numbers k we thus have

$$x_2^{(k-1)} = f(z) + \Delta_{k-1}, \text{ where } |\Delta_{k-1}| < \frac{2\varepsilon}{9}. \tag{3.7}$$

By (1.10), (3.7), for the function r_n [see (1.12)] in the interval $< a + + (i - 1)h, a + ih)$ we have:

$$r_n(x_1) = r_n^{(i)}(x_1) = \left(1, \left\langle\frac{x_1-a}{h}\right\rangle, \left\langle\frac{x_1-a}{h}\right\rangle^2, \langle\frac{x_1-a}{h}\rangle^3\right) \circ$$

$$\circ \frac{1}{16}\begin{bmatrix} -1 & 9 & 9 & -1 & 4m_1 & -4m_1 \\ 1 & -11 & 11 & -1 & -4m_1 & -4m_1 \\ 1 & -1 & -1 & 1 & -4m_1 & 4m_1 \\ -1 & 3 & -3 & 1 & 4m_1 & 4m_1 \end{bmatrix} \circ \begin{bmatrix} f(z)+\Delta_{i-1} \\ f(z)+\Delta_i \\ f(z)+\Delta_{i+1} \\ f(z)+\Delta_{i+2} \\ b_2^{(i)} \\ b_2^{(i+1)} \end{bmatrix} =$$

$$= \left(1, \left\langle\frac{x_1-a}{h}\right\rangle, \left\langle\frac{x_1-a}{h}\right\rangle^2, \left\langle\frac{x_1-a}{h}\right\rangle^3\right) \circ$$

$$\circ \frac{1}{16}\begin{bmatrix} 16f(z) - \Delta_{i-1} + 9\Delta_i + 9\Delta_{i+1} - \Delta_{i+2} + 4m_1 b_2^{(i)} - 4m_1 b_2^{(i+1)} \\ \Delta_{i-1} - 11\Delta_i + 11\Delta_{i+1} - \Delta_{i+2} - 4m_1 b_2^{(i)} - 4m_1 b_2^{(i+1)} \\ \Delta_{i-1} - \Delta_i - \Delta_{i+1} + \Delta_{i+2} - 4m_1 b_2^{(i)} + 4m_1 b_2^{(i+1)} \\ -\Delta_{i-1} + 3\Delta_i - 3\Delta_{i+1} + \Delta_{i+2} + 4m_1 b_2^{(i)} + 4m_1 b_2^{(i+1)} \end{bmatrix}.$$

For $x_1 = z$ we thus have

(3.8) $$r_n(z) - f(z) =$$

$$= \frac{1}{16}\Big\{ -\Delta_{i-1} + 9\Delta_i + 9\Delta_{i+1} - \Delta_{i+2} + 4m_1 b_2^{(i)} - 4m_1 b_2^{(i+1)} +$$
$$+ \left\langle\frac{z-a}{h}\right\rangle \Big[\Delta_{i-1} - 11\Delta_i + 11\Delta_{i+1} - \Delta_{i+2} - 4m_1 b_2^{(i)} - 4m_1 b_2^{(i+1)}\Big] +$$
$$+ \left\langle\frac{z-a}{h}\right\rangle^2 \Big[\Delta_{i-1} - \Delta_i - \Delta_{i+1} + \Delta_{i+2} - 4m_1 b_2^{(i)} + 4m_1 b_2^{(i+1)}\Big] +$$
$$+ \left\langle\frac{z-a}{h}\right\rangle^3 \Big[-\Delta_{i-1} + 3\Delta_i - 3\Delta_{i+1} + \Delta_{i+2} + 4m_1 b_2^{(i)} + 4m_1 b_2^{(i+1)}\Big]\Big\}.$$

Since $-1 \le \langle (z-a)/h\rangle < 1$ [see (1.9)], i.e.

(3.9) $$\left|\left\langle\frac{z-a}{h}\right\rangle\right| \le 1,$$

(3.8) implies, applying (3.6), (3.7), (3.9) that

(3.10) $$|r_n(z) - f(z)| < \frac{56}{16}\frac{2\varepsilon}{9} + \frac{32|m_1|}{16}\frac{\varepsilon}{9|m_1|} = \varepsilon$$

holds for all $n > n_0$ and arbitrary $z \in < a, b)$. Since for every $n \ge 2$ we have $r_n(b) = f(b)$ [see (1.12)], (3.10) is valid for all $n > n_0$ and arbitrary $z \in < a, b >$. Consequently, the following theorem holds.

Theorem. *Let f be a function continuous in the interval $< a, b >$ of finite length $L = b - a > 0$, and let $\varepsilon > 0$ be given. Then for the sequence*

$(r_n)_{n=2}$ *of* $L_{1,0}$*-splines* (1.12)*, each of which is constructed in the precedent sense for an arbitrary chosen number* $m_1 \neq 0$*, further for arbitrary chosen numbers* $|z_2| \leq 8\varepsilon/9, |u_2| \leq 8\varepsilon/9, |c_2| \leq 8|m_1|, |d_2| \leq 8|m_1|$ *[see* (1.4) *for* $j = 2$*] and arbitrary chosen values* $x_2^{(0)}, x_2^{(n+2)}, x_2^{(n+3)}$ *in* (3.3)*,* (3.10) *is valid for sufficiently large* n *and arbitrary* $z \in< a, b >$*, i.e. the sequence converges uniformly to the function* f *in the interval* $<< a, b >$*. In other words, for almost all* n *the mentioned* $L_{1,0}$*-spline approximations* $l_n = \{(x_1, x_2) \in R^2 | a \leq x_1 \leq b, x_2 = r_n(x_1)\}$ *of the curve* $l = \{(x_1, x_2) \in R^2 | a \leq x_1 \leq b, x_2 = f(x_1)\}$ *lie in its Euclidean neighbourhood with diameter* 2ε*.*

Analogous conclusion can be derived for a case of continuous vector function $f == (f_1, f_2, \ldots, f_{m-1}) :< a, b >\rightarrow R^{m-1}, m > 2$ integer.

Example. Consider the function $x_2 = f(x_1) = 0.001x_1^3 + x_1$ in the interval $< 0, 10 >, L = 10$. For $x_1', x_1'' \in< 0, 10 >$ we have $|f(x_1') - - f(x_1'')| = |x_1' - x_1''||0.001[x_1'^2 + +x_1'x_1'' + x_1''^2] + 1| \leq (1.3)|x_1' - x_1''|$; for $|x_1' - x_1''| < \delta = (2\varepsilon/9)(1.3)^{-1}$ we then have $|f(x_1') - f(x_1'')| < 2\varepsilon/9$ [see (3.1)]. Then for the sequence $(r_n)_{n=2}$ of $L_{1,0}$-splines (1.12), each of which is constructed in the sense of the derived theorem, we have

$$|r_n(z) - f(z)| < \varepsilon \tag{3.11}$$

for all $n > n_0 = \max\{1, [30/\delta] + 1\}$ [see (3.2)] and arbitrary $z \in< < 0, 10 >$. For instance, for $\varepsilon = 0.9$ we have $n_0 = 196$. Since the estimate (3.10) is in a sense "rough", we may expect inequality (3.11) to hold for all $z \in< 0, 10 >$ for substantially smaller $n_0 \geq 1$.

References

[1] MATUŠŮ and J., NOVÁK, J.: A method for construction of spline curves, *Mathematica Pannonica,* **1**/2 (1990), 7–23.

[2] MATUŠŮ, J., DOHNAL, G. and MATUŠŮ, M.: On one method of numerical integration, *Applications of Mathematics,* **36**/4 (1991), 241–263.

[3] LIENHARD, H.: Interpolation von Funktionswerten bei numerischen Bahnsteuerungen, *Undated publication of CONTRAVES AG, Zürich,* 1–7.

[4] MATUŠŮ, J.: The Lienhard interpolation and some of its generalizations (in Czech), *Acta Polytechnica - Prague* **3–IV** (1978), 5–31.