Mathematica Pannonica
6/2 (1995), 217 – 228

TUBULAR NURB SURFACES WITH BOUNDARY CONTROL

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Dedicated to Professor Hans Vogler on the occasion of his 60th birthday

Received October 1994

MSC 1991: 41 A 15, 65 Y 25, 68 U 05, 68 U 07

Keywords: CAGD, NURB surfaces, B-splines.

Abstract: The presented algorithm generates tube shaped NURB surfaces with given boundary conditions. A generalization of the technique of phantom vertices of B-spline curves is given for surface design. The proposed method is also suitable for the tangential fitting of surfaces.

Introduction

In surface design a crucial problem is the planning of boundary curves according to in part freely chosen boundary conditions that influence the shape of the surface or ensure its smooth joining to another surface. For both problems several solutions have been developed in NURB (non-uniform rational B-splines) technique. The shape design has been solved mostly by choosing appropriate knot vectors or multiple control vertices [2,3], or additional shape parameters [1], and the fitting problem has been solved by constructing appropriate blending surfaces [6,7]. The presented algorithm also uses NURB functions and is suitable for the solution of both problems.

Two computational methods of the NURB technique form the basis of the present algorithm. The first is the application of phantom

Supported by the Hung. Nat. Found. for Sci. Research (OTKA) grant no. 1615 (1991)

vertices determined from given endpoints, tangents and curvatures of B-spline curves [5]. The second is the matrix representation of NURB curves and surfaces, giving a convenient technique for the computation of periodic B-splines [4].

The present algorithm generates tube shaped NURB surfaces with given boundary conditions. For the boundary control the following input data are allowed: interpolation points for the closed curves at both ends of the tube and the tangent directions of the longitudinal parameter lines of the surface at the same points. In this way the generated tubular surface can be shaped easily at the ends. In the case when a tube has to be fitted to another surface like a handle of a flacon, the interpolation points and the corresponding derivatives are to be computed from the function describing the other surface. According to the given or computed boundary conditions a set of control vertices are determined, the so called *phantom points* that are invisible for the user. At the inner points of the surface the usual manipulations of NURBs by control vertices and weights are allowed.

Underlying technique

In this paragraph a summary of the computation of one-parametric NURB splines is given, particularly that of cubic and closed quadratic curves. In particular, such curves are the longitudinal and cross section parameter lines of the generated tube shaped surfaces.

As it is well-known, a cubic NURB curve segment over the parameter interval $[0,1]$ has the following matrix form:

$$\mathbf{r}_i(u) = \frac{\mathbf{R}_i(u)}{h_i(u)}, \qquad u \in [0,1],$$

for the ith interval $(i = 1, 2, \dots, n-2)$. Here $\mathbf{R}_i(u) = \mathbf{U}\mathbf{N}_i\mathbf{V}$, $h(u) = \mathbf{U}\mathbf{N}_i\mathbf{W}$, $\mathbf{U} = [1 \quad u \quad u^2 \quad u^3]$,

$$\mathbf{V} = [w_i\mathbf{V}_i \quad w_{i+1}\mathbf{V}_{i+1} \quad w_{i+2}\mathbf{V}_{i+2} \quad w_{i+3}\mathbf{V}_{i+3}]^T,$$

where $\mathbf{V}_i$, $\mathbf{V}_{i+1}$, $\mathbf{V}_{i+2}$, $\mathbf{V}_{i+3}$ denote the control vertices influencing the ith segment, $\mathbf{W} = [w_i \quad w_{i+1} \quad w_{i+2} \quad w_{i+3}]^T$, $w_i > 0$ scalar values are the weights of the control vertices, $\{t_1, t_2, \dots, t_{n+5}\}$ is the knot vector, which is a non-decreasing sequence of the knot points $\{t_i\}$ (real numbers) and $u = (t - t_i)/(t_{i+1} - t_i)$, $t \in [t_i, t_{i+1}]$, $\mathbf{N}_i = [n_{rc}]_i$ is a 4×4 coefficient matrix determined by the knot vector that has the following elements (cf. [4]):

$$\mathbf{N}_i = \begin{bmatrix} n_{11} & 1 - n_{11} - n_{13} & n_{13} & 0 \\ -3n_{11} & 3n_{11} - n_{23} & n_{23} & 0 \\ 3n_{11} & -(3n_{11} + n_{33}) & n_{33} & 0 \\ -n_{11} & n_{11} - n_{43} - n_{44} & n_{43} & n_{44} \end{bmatrix},$$

where

$$n_{11} = \frac{(t_{i+5} - t_{i+4})^2}{(t_{i+5} - t_{i+3})(t_{i+5} - t_{i+2})}, \quad n_{13} = \frac{(t_{i+4} - t_{i+3})^2}{(t_{i+6} - t_{i+3})(t_{i+5} - t_{i+3})},$$

$$n_{23} = \frac{3(t_{i+5} - t_{i+4})(t_{i+4} - t_{i+3})}{(t_{i+6} - t_{i+3})(t_{i+5} - t_{i+3})}, \quad n_{33} = \frac{3(t_{i+5} - t_{i+4})^2}{(t_{i+6} - t_{i+3})(t_{i+5} - t_{i+3})},$$

$$n_{43} = -\left\{\frac{1}{3}n_{33} + n_{44} + \frac{(t_{i+5} - t_{i+4})^2}{(t_{i+6} - t_{i+4})(t_{i+6} - t_{i+3})}\right\},$$

$$n_{44} = \frac{(t_{i+5} - t_{i+4})^2}{(t_{i+7} - t_{i+4})(t_{i+6} - t_{i+4})},$$

and $\nabla_i = t_{i+4} - t_{i+3} = \lambda|\mathbf{V}_i\mathbf{V}_{i+1}|$, which is the chord-length parametrization. In the uniform case the usual choice is $t_i = i$.

The matrices $\mathbf{N}_i$ of a uniform B-spline do not depend on i, i.e. for each curve segment

$$\mathbf{N}_i = \mathbf{N} = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

When $w_i = 1$ for all i, the denominators $h_i(u)$ of $\mathbf{r}_i(u)$ are identically 1, consequently the vector valued functions $\mathbf{r}_i(u)$ are polynomials (non rational). The whole cubic NURB spline over the knot vector consists of $n - 2$ segments and is C^2-continuous everywhere, if there are no multiple control vertices. The number of control vertices, therefore the number of weights is $n + 1$. The following scheme shows the indices of the knot points, control vertices and intervals used in this paper.

			$\mathbf{V}_1$	$\mathbf{V}_2$			$\mathbf{V}_n$	$\mathbf{V}_{n+1}$	
t_1	t_2	t_3	t_4	t_5	t_{n+1}	t_{n+2}	t_{n+3}	t_{n+4}	t_{n+5}
				∇_1	∇_{n-3}	∇_{n-2}			

The starting point $\mathbf{P}_s$ and the endpoint $\mathbf{P}_e$ of the curve are given by the vectors $\mathbf{r}_1(0)$ and $\mathbf{r}_{n-2}(1)$, respectively. The tangent vectors are given by the values of the first derivatives at the same places, where

$$\dot{\mathbf{r}}(u) = \mathbf{R}(u)\left[-\frac{\dot{h}(u)}{h^2(u)}\right] + \dot{\mathbf{R}}(u)\frac{1}{h(u)}.$$

In the following the method of phantom points determined by boundary conditions is summarized, which is well-known for one-parametric B-spline curves. The proof of the following lemma in the polynomial case is given, for example, in [5].

Lemma. *If the starting point $\mathbf{P}_s$ of a periodic* NURB *curve and the tangent vector $\mathbf{T}_s$ at this point are given, then the control points $\mathbf{V}_1$ and $\mathbf{V}_2$ of weights 1 are determined as the solution of a system of linear equations and, similarly, the last two control points $\mathbf{V}_n$ and $\mathbf{V}_{n+1}$ of weights 1 can be computed from the given endpoint $\mathbf{P}_e$ and the tangent vector $\mathbf{T}_e$ at that point.*

Proof. Let the knot vector and the control vertices $\mathbf{V}_3, \dots, \mathbf{V}_{n-1}$, furthermore the weights $w_1 = w_2 = 1, w_3, \dots, w_{n-1}, w_n = w_{n+1} = 1$ be given. The equation of the ith curve segment is

$$\mathbf{r}_i(u) = \frac{\mathbf{R}(u)}{h(u)}, \quad 0 \le u \le 1, \quad i = 1, \dots, n-2.$$

The data

$$\mathbf{P}_s = \mathbf{r}_1(0), \ \mathbf{T}_s = \dot{\mathbf{r}}_1(0), \ \mathbf{P}_e = \mathbf{r}_{n-2}(1), \ \mathbf{T}_e = \dot{\mathbf{r}}_{n-2}(1)$$

are also given. For the first segment of the curve the value of $\mathbf{R}(u)$ and $\dot{\mathbf{R}}(u)$ at $u = 0$ can be computed as follows:

$$\begin{aligned}
\mathbf{R}(0) &= [1 \quad 0 \quad 0 \quad 0]\,[n_{rc}]_1\,[\mathbf{V}_1 \quad \mathbf{V}_2 \quad w_3\mathbf{V}_3 \quad w_4\mathbf{V}_4]^T \\
&= [n_{11} \quad n_{12} \quad n_{13} \quad 0]\,[\mathbf{V}_1 \quad \mathbf{V}_2 \quad w_3\mathbf{V}_3 \quad w_4\mathbf{V}_4]^T \\
\dot{\mathbf{R}}(0) &= [0 \quad 1 \quad 0 \quad 0]\,[n_{rc}]_1\,[\mathbf{V}_1 \quad \mathbf{V}_2 \quad w_3\mathbf{V}_3 \quad w_4\mathbf{V}_4]^T \\
&= [n_{21} \quad n_{22} \quad n_{23} \quad 0]\,[\mathbf{V}_1 \quad \mathbf{V}_2 \quad w_3\mathbf{V}_3 \quad w_4\mathbf{V}_4]^T.
\end{aligned}$$

Similarly, at the endpoint of the $(n-2)$th segment

$$\begin{aligned}
\mathbf{R}(1) &= [1 \quad 1 \quad 1 \quad 1][n_{rc}]_{n-2}[w_{n-2}\mathbf{V}_{n-2} \quad w_{n-1}\mathbf{V}_{n-1} \quad \mathbf{V}_n \quad \mathbf{V}_{n+1}]^T, \\
\dot{\mathbf{R}}(1) &= [0 \quad 1 \quad 2 \quad 3][n_{rc}]_{n-2}[w_{n-2}\mathbf{V}_{n-2} \quad w_{n-1}\mathbf{V}_{n-1} \quad \mathbf{V}_n \quad \mathbf{V}_{n+1}]^T.
\end{aligned}$$

The corresponding expressions for the denominator $h(u)$ of $\mathbf{r}(u)$ differ only in the last row matrix, which contains only the scalar weights of the components. Consequently,

$$\mathbf{r}_1(0) = \frac{n_{11}\mathbf{V}_1 + n_{12}\mathbf{V}_2 + n_{13}w_3\mathbf{V}_3}{n_{11} + n_{12} + n_{13}w_3} = \mathbf{P}_s,$$

$$\dot{\mathbf{r}}_1(0) = \frac{-(n_{21} + n_{22} + n_{23}w_3)}{(n_{11} + n_{12} + n_{13}w_3)^2}(n_{11}\mathbf{V}_1 + n_{12}\mathbf{V}_2 + n_{13}w_3\mathbf{V}_3) +$$

$$+ \frac{1}{n_{11} + n_{12} + n_{13}w_3}(n_{21}\mathbf{V}_1 + n_{22}\mathbf{V}_2 + n_{23}w_3\mathbf{V}_3) = \mathbf{T}_s,$$

which is a system of linear equations of the following structure:

$$\mathbf{P}_s + b_1\mathbf{V}_3 = a_{11}\mathbf{V}_1 + a_{12}\mathbf{V}_2$$

$$\mathbf{T}_s + b_2\mathbf{V}_3 = a_{21}\mathbf{V}_1 + a_{22}\mathbf{V}_2.$$

The unknowns are the control points $\mathbf{V}_1$ and $\mathbf{V}_2$, the scalar coefficients are given by the elements of $[n_{rc}]_1$ and the weight w_3. If the determinant of the system is zero, either a knot point or the control vertex $\mathbf{V}_3$ or its weight w_3 has to be changed. In this way the system can be made regular. In particular, in the case of uniform polynomial B-spline this system reduces to

$$\mathbf{P}_s - \frac{1}{6}\mathbf{V}_3 = \frac{1}{6}\mathbf{V}_1 + \frac{4}{6}\mathbf{V}_2$$

$$\mathbf{T}_s - \frac{1}{2}\mathbf{V}_3 = -\frac{1}{2}\mathbf{V}_1,$$

which always has a unique solution $(\mathbf{V}_1, \mathbf{V}_2)$. The corresponding system of linear equations at the endpoint has the form

$$\mathbf{P}_e + d_1\mathbf{V}_{n-2} + e_1\mathbf{V}_{n-1} = b_{11}\mathbf{V}_n + b_{12}\mathbf{V}_{n+1}$$

$$\mathbf{T}_e + d_2\mathbf{V}_{n-2} + e_2\mathbf{V}_{n-1} = b_{21}\mathbf{V}_n + b_{22}\mathbf{V}_{n+1},$$

where the coefficients d_i, e_i, b_{ij} are scalars given by $[n_{rc}]_{n-2}$, w_{n-2} and w_{n-1}. In particular, in the uniform polynomial case

$$\mathbf{P}_e - \frac{1}{6}\mathbf{V}_{n-1} = \frac{4}{6}\mathbf{V}_n + \frac{1}{6}\mathbf{V}_{n+1}$$

$$\mathbf{T}_e + \frac{1}{2}\mathbf{V}_{n-1} = \frac{1}{2}\mathbf{V}_{n+1},$$

which always has a unique solution $(\mathbf{V}_n, \mathbf{V}_{n+1})$. $\Diamond$

The vertices $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_n$, $\mathbf{V}_{n+1}$ determined by the end conditions are called *phantom vertices*, unknown and unvisible for the user. Therefore the assumption for the corresponding weights $w_1 = w_2 = = w_{n-2} = w_{n-1} = 1$ does not limit the possibilities of shape control.

A periodic closed quadratic NURB curve is determined in a similar way by the knot vector $s_1 < s_2 < \ldots < s_{m+4}$, the control vertices $\mathbf{V}_1 \equiv \mathbf{V}_m$, $\mathbf{V}_2 \equiv \mathbf{V}_{m+1}$, $\mathbf{V}_3, \ldots, \mathbf{V}_{m-1}$ and the corresponding weights

$w_1, \dots, w_{m-1}$. The curve consists of $m-1$ segments, and its first derivative is continuous at each point if there are no multiple vertices. The indices of the knot points and control vertices are shown in the following sketch

$$\mathbf{V}_1 \quad \mathbf{V}_2 \qquad\qquad \mathbf{V}_m \equiv \mathbf{V}_1 \quad \mathbf{V}_{m+1} \equiv \mathbf{V}_2$$

$$s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \qquad\qquad s_m \quad s_{m+1} \quad s_{m+2} \quad s_{m+3} \quad s_{m+4}$$

$$\triangle_1 \quad \triangle_2 \qquad\qquad \triangle_{m-2} \quad \triangle_{m-1} \qquad\qquad \triangle_1$$

Each segment is parametrized from 0 to 1 and is represented by the vector function

$$\mathbf{r}_i(u) = \frac{[1 \quad u \quad u^2][m_{rc}]_i [w_i \mathbf{V}_i \quad w_{i+} \mathbf{V}_{i+1} \quad w_{i+2} \mathbf{V}_{i+2}]^T}{[1 \quad u \quad u^2][m_{rc}]_i [w_i \quad w_{i+1} \quad w_{i+2}]^T}$$

$(i = 1, \dots, m-1)$, where

$$u = (s - s_i)/(s_{i+1} - s_i) = (s - s_i)/\triangle_i, \quad s \in [s_i, s_{i+1}].$$

The elements of the matrix determined by the knot vector are:

$$[m_{rc}]_i = \begin{bmatrix} m_{11} & m_{12} & 0 \\ -2m_{11} & 2m_{11} & 0 \\ m_{11} & m_{32} & m_{33} \end{bmatrix},$$

$$m_{11} = \frac{s_{i+3} - s_{i+2}}{s_{i+3} - s_{i+1}}, \quad m_{12} = \frac{s_{i+2} - s_{i+1}}{s_{i+3} - s_{i+1}},$$

$$m_{32} = -(s_{i+3} - s_{i+2}) \left[\frac{1}{s_{i+3} - s_{i+1}} + \frac{1}{s_{i+4} - s_{i+2}} \right], \; m_{33} = \frac{s_{i+3} - s_{i+2}}{s_{i+4} - s_{i+2}}.$$

In the chord-length parametrization we have $\triangle_i = s_{i+3} - s_{i+2} = \lambda |\mathbf{V}_i \mathbf{V}_{i+1}|$. In the uniform case we obtain

$$[m_{rc}] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

Boundary conditions for tubular surfaces

In this paragraph the technique of boundary control applying phantom points is shown for two-parametric spline surfaces what is a generalization of the case of one-parametric curves. The algorithm for the boundary control of tube shaped surfaces is developed for two-parametric NURB representation of 3×2 degrees. The u-parameter lines of the $\mathbf{r}(u,v)$ vector function are rational cubic longitudinal curves, and the closed cross-section curves of the surface (which require less elasticity in the planning process) are quadratic rational functions of the second parameter v. A patch of the composed surface is described by

$$\mathbf{r}_{ij}(u,v) = \frac{\mathbf{R}(u,v)}{h(u,v)},$$

$$0 \le u \le 1,\ 0 \le v \le 1,\ i = 1, \dots n-2,\ j = 1, \dots, m-1,$$

where

$$\mathbf{R}(u,v) = [1 \quad u \quad u^2 \quad u^3\,][\,n_{rc}\,]_i\,[\,w_{ij}\mathbf{V}_{ij}\,][\,m_{rc}\,]_j^T\,[1 \quad v \quad v^2\,]^T,$$

$$h(u,v) = [1 \quad u \quad u^2 \quad u^3\,][\,n_{rc}\,]_i\,[\,w_{ij}\,][\,m_{rc}\,]_j^T\,[1 \quad v \quad v^2\,]^T,$$

and

$$\mathbf{V}_{ij} = \begin{bmatrix} \mathbf{V}_{i,j} & \mathbf{V}_{i,j+1} & \mathbf{V}_{i,j+2} \\ \mathbf{V}_{i+1,j} & \mathbf{V}_{i+1,j+1} & \mathbf{V}_{i+1,j+2} \\ \mathbf{V}_{i+2,j} & \mathbf{V}_{i+2,j+1} & \mathbf{V}_{i+2,j+2} \\ \mathbf{V}_{i+3,j} & \mathbf{V}_{i+3,j+1} & \mathbf{V}_{i+3,j+2} \end{bmatrix}$$

is the matrix of control vertices influencing the (i,j)th patch and $w_{i,j}$ are the corresponding weights. $[\,n_{rc}\,]_i$ is the cubic B-spline coefficient matrix determined by the knot vector $\{t_1, \dots, t_{n+5}\}$, $[\,m_{rc}\,]_j$ is the quadratic B-spline coefficient matrix determined by the knot vector $\{s_1, \dots, s_{m+4}\}$.

$$u = (t - t_i)/(t_{i+1} - t_i), \qquad t \in [t_i, t_{i+1}], \quad i = 1, \dots, n+4,$$

$$v = (s - s_j)/(s_{j+1} - s_j), \quad s \in [s_j, s_{j+1}] \quad j = 1, \dots, m+3.$$

The total number of control vertices is $(n+1) \times (m+1)$. For the control points of the closed cross section v-parameter curves $\mathbf{V}_{i,m-1} = \mathbf{V}_{i,1}$; $\mathbf{V}_{i,m} = \mathbf{V}_{i,1}$; $\mathbf{V}_{i,m+1} = \mathbf{V}_{i,2}$ must hold. The derivatives

$$\mathbf{r}_u = \frac{\mathbf{R}_u h - \mathbf{R} h_u}{h^2} \quad \text{and} \quad \mathbf{r}_v = \frac{\mathbf{R}_v h - \mathbf{R} h_v}{h^2}$$

determine the tangent plane of the surface at each point. These formulas become rather simple at the corner points of the patch, i.e. for the parameter values $u = 0, 1$ and $v = 0, 1$. For the patches, whose control points are of weights 1, the vector function $\mathbf{r}_{ij}(u,v)$ is polynomial, and the derivatives are easier to compute.

Consider the boundary curves of the tube shaped surface. The jth segments of the two curves are given by $\mathbf{r}_{1j}(0,v)$ and $\mathbf{r}_{n-2,j}(1,v)$ $j = 1, \dots, m-1$, respectively. In the planning process the problem arises,

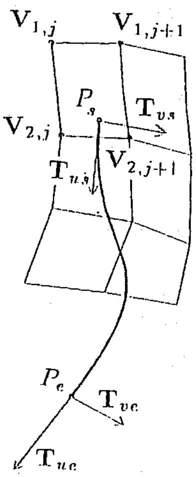

Figure 1. Control points of a patch depending on the boundary conditions.

how the boundaries can be forced to pass through given points, at which the tangent plane of the surface is also given. For the sake of simplicity the segmentation and parametrization of the boundary curves will be chosen in such a way that the given interpolation points are the corner points $\mathbf{r}_{1,j}(0,0)$ and $\mathbf{r}_{n-2,j}(1,0)$ $(j = 1,3,5,\ldots,m-1)$ of the patches at the ends of the tube, and the tangent planes and the twist vectors are given at these points as well. The following theorem shows that four control vertices of each patch along a boundary curve can be determined from the boundary conditions. These vertices are the so called *phantom points* and are determined as the solution of a 4×4 linear equation system at both ends of the tube.

Theorem. *The corner point $\mathbf{P}_s$, the derivatives $\mathbf{T}_{us}, \mathbf{T}_{vs}$ and the twist vector $\mathbf{T}_{uvs}$ at the point $(u,v) = (0,0)$ determine the control vertices $\mathbf{V}_{1,j}$, $\mathbf{V}_{2,j}$, $\mathbf{V}_{1,j+1}$, $\mathbf{V}_{2,j+1}$ of the patch $\mathbf{r}_{1,j}(u,v)$ uniquely. Similarly, the control vertices $\mathbf{V}_{n,j}$, $\mathbf{V}_{n+1,j}$, $\mathbf{V}_{n,j+1}$ $\mathbf{V}_{n+1,j+1}$ of the patch $\mathbf{r}_{n-2,j}(u,v)$ are determined by the corner point $\mathbf{P}_e$, the derivatives $\mathbf{T}_{ue}$, $\mathbf{T}_{ve}$ and the twist vector $\mathbf{T}_{uve}$ at the point $(u,v) = (1,0)$ $(j = 1,3\ldots,m-1)$.*

Proof. For simplicity the computation will be carried out for the polynomial case. It can be seen from the proof of the Lemma that the structure of the equations in the rational case is the same, because the denominators contain only constant scalar values. By assumption, the control points of the bordering patches $\mathbf{r}_{1,j}(u,v)$ and $\mathbf{r}_{n-2,j}(u,v)$ are simple vertices and their weights are set to be 1. The boundary conditions along the upper end of the tube (Fig. 1) give the following equations for the patch $\mathbf{r}_{1,j}(u,v)$:

$$\mathbf{P}_s = \mathbf{r}_{1,j}(0,0) = [1 \quad 0 \quad 0 \quad 0]\,[n_{rc}]_i\,[\mathbf{V}_{ij}]\,[m_{rc}]_j^T\,[1 \quad 0 \quad 0]^T$$
$$= [n_{11} \quad n_{12} \quad n_{13} \quad n_{14}]\,[\mathbf{V}_{ij}]\,[m_{11} \quad m_{21} \quad m_{31}]^T,$$
$$\mathbf{T}_{us} = \frac{\partial}{\partial u}\mathbf{r}_{1j}(0,0) = [0 \quad 1 \quad 0 \quad 0]\,[n_{rc}]_i\,[\mathbf{V}_{ij}]\,[m_{rc}]_j^T\,[1 \quad 0 \quad 0]^T$$

$$= [n_{21} \quad n_{22} \quad n_{23} \quad n_{24}][\mathbf{V}_{ij}][m_{11} \quad m_{21} \quad m_{31}]^T,$$

$$\mathbf{T}_{vs} = \frac{\partial}{\partial v}\mathbf{r}_{1j}(0,0) = [1 \quad 0 \quad 0 \quad 0][n_{rc}]_i[\mathbf{V}_{ij}][m_{rc}]_j^T[0 \quad 1 \quad 0]^T$$

$$= [n_{11} \quad n_{12} \quad n_{13} \quad n_{14}][\mathbf{V}_{ij}][m_{12} \quad m_{22} \quad m_{32}]^T,$$

$$\mathbf{T}_{uvs} = \frac{\partial}{\partial u}\frac{\partial}{\partial v}\mathbf{r}_{1j}(0,0) = [0 \quad 1 \quad 0 \quad 0][n_{rc}]_i[\mathbf{V}_{ij}][m_{rc}]_j^T[0 \quad 1 \quad 0]^T$$

$$= [n_{21} \quad n_{22} \quad n_{23} \quad n_{24}][\mathbf{V}_{ij}][m_{12} \quad m_{22} \quad m_{32}]^T$$

In the uniform case, substituting the elements of the constant matrices N and M, the equations are the following:

$$\mathbf{V}_{1,j} + 4\mathbf{V}_{2,j} + \mathbf{V}_{1,j+1} + 4\mathbf{V}_{2,j+1} = 12\mathbf{P}_s - \mathbf{V}_{3,j} - \mathbf{V}_{3,j+1},$$

$$-3\mathbf{V}_{1,j} - 3\mathbf{V}_{1,j+1} = 12\mathbf{T}_{us} - 3\mathbf{V}_{3,j} - 3\mathbf{V}_{3,j+1},$$

$$-2\mathbf{V}_{1,j} - 8\mathbf{V}_{2,j} + 2\mathbf{V}_{1,j+1} + 8\mathbf{V}_{2,j+1} = 12\mathbf{T}_{vs} + 2\mathbf{V}_{3,j} - 2\mathbf{V}_{3,j+1},$$

$$6\mathbf{V}_{1,j} - 6\mathbf{V}_{1,j+1} = 12\mathbf{T}_{uvs} + 6\mathbf{V}_{3,j} - 6\mathbf{V}_{3,j+1}.$$

This system of equations has a unique solution for the control vertices $\mathbf{V}_{1,j}$, $\mathbf{V}_{1,j+1}$, $\mathbf{V}_{2,j}$, $\mathbf{V}_{2,j+1}$, $(j = 1, 3 \ldots, m-1)$. The equations in the non-uniform case differ only in the coefficients. If the determinent of the system is zero, either a knot point or a control vertex ($\mathbf{V}_{3,j}$ or $\mathbf{V}_{3,j+1}$) has to be changed a bit. The corresponding system of equations at the other end of the tube for the patches $\mathbf{r}_{n-2,j}(u,v)$, $(j = 1, 3 \ldots, m-1)$ in the uniform case can be obtained from the conditions

$$\mathbf{P}_e = \mathbf{r}_{n-2,j}(1,0), \qquad \mathbf{T}_{ue} = \frac{\partial}{\partial u}\mathbf{r}_{n-2,j}(1,0),$$

$$\mathbf{T}_{ve} = \frac{\partial}{\partial v}\mathbf{r}_{n-2,j}(1,0), \quad \mathbf{T}_{uve} = \frac{\partial}{\partial u}\frac{\partial}{\partial v}\mathbf{r}_{n-2,j}(1,0),$$

and is the following:

$$\mathbf{V}_{n,j} + \mathbf{V}_{n+1,j} + 4\mathbf{V}_{n,j+1} + \mathbf{V}_{n+1,j+1} = 12\mathbf{P}_e - \mathbf{V}_{n-1,j} - \mathbf{V}_{n-1,j+1},$$

$$3\mathbf{V}_{n+1,j} + 3\mathbf{V}_{n+1,j+1} = 12\mathbf{T}_{ue} + 3\mathbf{V}_{n-1,j} + 3\mathbf{V}_{n-1,j+1},$$

$$-8\mathbf{V}_{n,j} - 2\mathbf{V}_{n+1,j} + 8\mathbf{V}_{n,j+1} + 2\mathbf{V}_{n+1,j+1} = 12\mathbf{T}_{ve} + 2\mathbf{V}_{n-1,j} - 2\mathbf{V}_{n-1,j+1},$$

$$-6\mathbf{V}_{n+1,j} + 6\mathbf{V}_{n+1,j+1} = 12\mathbf{T}_{uve} - 6\mathbf{V}_{n-1,j} + 6\mathbf{V}_{n-1,j+1}.$$

This system has a unique solution for the control points $\mathbf{V}_{n,j}$, $\mathbf{V}_{n+1,j}$, $\mathbf{V}_{n,j+1}$ and $\mathbf{V}_{n+1,j+1}$ $(j = 1, 3 \ldots, m-1)$. This fact shows that if the determinant of the system in the non-uniform case is zero, a change either of the knot vector or of a control vertex ($\mathbf{V}_{n-1,j}$ or $\mathbf{V}_{n-1,j+1}$) makes the system regular. The proof of the theorem is complete. ◊

As the control points in the jth and $j+1$th columns are determined by the boundary conditions at the corner point of the jth patch, the computation of phantom points has to be carried out for each second patch along the boundary curves. The assumption on the weights at the bordering patches is in practice a very small restriction. For, the phantom vertices cannot be changed by the user, consequently their weights can be fixed. In this way the assumptions $w_{1,j} = w_{1,j+1} = w_{2,j} = w_{2,j+1} = 1$ and $w_{n,j} = w_{n,j+1} = w_{n+1,j} = w_{n+1,j+1} = 1$ do not mean any restriction in the planning process, only the conditions $w_{3,j} = w_{3,j+1} = w_{4,j} = w_{4,j+1} = 1$ and $w_{n-1,j} = w_{n-1,j+1} = w_{n,j} = w_{n,j+1} = 1$ limit the freedom of the user. The control vertices $\mathbf{V}_{4,j}$, $\mathbf{V}_{4,j+1}, \dots, \mathbf{V}_{n-2,j}, \mathbf{V}_{n-2,j+1}$ and the weights $w_{5,j}, w_{5,j+1}, \dots, w_{n-3,j}$, $w_{n-3,j+1}$ $(j = 1, 3 \dots, m-1)$ do not influence the shape along the end curves, and they can be manipulated as usual in NURB technique.

Applying rational vector functions $\mathbf{r}_{1,j}(u,v)$ and $\mathbf{r}_{n-2,j}(u,v)$ along the two boundaries, the limitation made for the weights can be lifted and the structure of the equations for the phantom vertices remains the same, because the denominators contain given scalar values only. Consequently, the proof of the theorem in the rational case is practically the same.

Remark. The third row of the matrix $\mathbf{V}_{ij}$ does not influence the phantom vertices determined by the endpoints of the longitudinal u-parameter curves, the first derivatives and the twist vectors at these points. That makes possible to split the surface into longitudinal stripes formed by two columns of patches. In such a way one interpolation point and the corresponding derivatives, which will determine four control points, can be prescribed at both ends of each stripe.

The set of input data standing on the right-hand side of the system of equations can be reduced by computing the derivatives in the v-parameter direction from the given interpolation points and setting the twist vectors to be zero. The remaining tangent vector in the longitudinal direction will shape the surface easily along its boundary.

Examples. In Fig. 2 a tubular NURB surface is shown defined by the control net, which is drawn by broken lines. If the tangents of the longitudinal lines and the tangents of the boundary curves are changed at the corner points $(u,v) = (0,0)$ of the upper bordering patches and at $(u,v) = (1,0)$ of the lower bordering patches (Fig. 3), then two rows of control vertices at both ends of the tube will change. The corresponding control net is shown in Fig. 4.

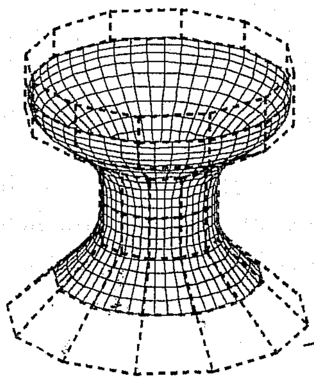

Figure 2. Periodic NURB surface and the net of control vertices.

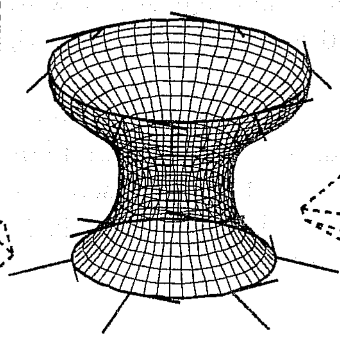

Figure 3. NURB surface generated with the help of given tangents at boundary points.

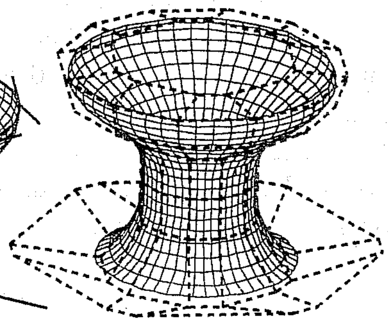

Figure 4. Surface and the control net computed from given boundary conditions.

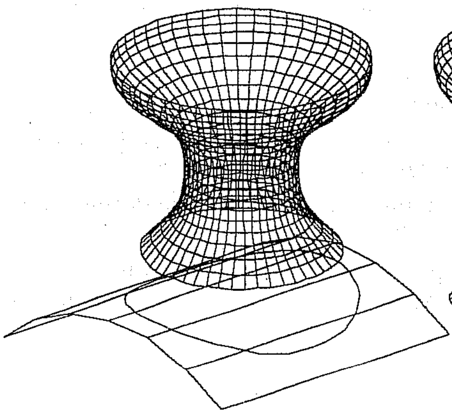

Figure 5. First order continuous fitting; the lower boundary, tangents and twist-vectors will be replaced.

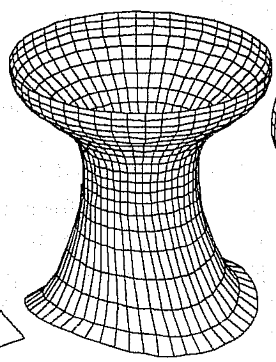

Figure 6. Surface fitted to a cylindrical surface along a given curve.

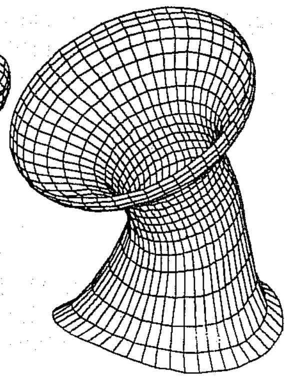

Figure 7. Rotated surface with the same fitting.

A surface fitting problem can be seen in Fig. 5. The tube will be fitted to a cylinder along a given surface curve in such a way that the new lower boundary line of the tube interpolates the given curve, and the tangent planes and twist vectors of the two surfaces are common at the interpolation points. The generated surfaces with the new boundary curve are shown in Fig. 6 and Fig. 7 in the original and in a rotated position, respectively.

Conclusions. A method for generating tubular NURB surfaces with given closed boundary curves and tangent planes at their points has been presented in this paper. The proposed algorithm allows the controlling the shape of surfaces along the boundaries and solves the fitting problem of such surfaces with tangent plane continuity. The necessary computations are organized in such a way that they can be implemented easily in computer programs for surface design.

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