Mathematica Pannonica
6/2 (1995), 175 – 180

ON GEODESIC MAPPINGS OF RECURRENT FINSLER SPACES

Sándor **BÁCSÓ**

Department of Geometry, Kossuth Lajos University, H–4010 Debrecen, P. O. Box 12, Hungary

Ildikó **PAPP**

Chair in Mathematics and Descriptive Geometry, Department of Basic Studies, Ybl Miklós Institute Polytechnic, H–4011 Debrecen, P. O. Box 40, Hungary

Dedicated to Professor Hans Vogler on his 60th birthday

Received November 1994

MSC 1991: 53 B 40

Keywords: Finsler geometry, geodesic mapping.

Abstract: The aim of the present paper is to study some properties in recurrent Finsler spaces. First we prove that a recurrent Finsler space of nonzero scalar curvature is a Riemannian space of nonzero constant curvature. Further on we will investigate the geodesic mappings of special Finsler spaces into a recurrent Finsler space.

1. Introduction

Let $F^n(M^n, L)$ be an n-dimensional Finsler space, where M^n is a connected differentable manifold of dimension n and $L(x, y)$, where $y^i = \dot{x}^i$ [1)], is the fundamental function defined on the manifold $TM\backslash 0$

This work was partially supported by the Hungarian Foundation for Scientific Research under grant number OTKA-1653.

1) The Roman and Greek indices run over the range $1, \ldots, n$; the Roman indices are free but the Greek indices denote summation.

of nonzero tangent vectors. We assume that $L(x,y)$ is positive and the metric tensor $g_{ij}(x,y) = \frac{1}{2}L^2_{(i)(j)}$, ($_{(i)} = \partial/\partial y^i$) is positiv definit.

The Berwald connection $B\Gamma = (G^i_j, G^i_{jk})$ of a Finsler space is constructed from the function $G^i(x,y)$ appearing in the equation of the geodesics

$$d^2x^i/dt^2 + 2G^i(x,y) = 0.$$

The $G^i(x,y)$ are positively homogeneous functions of degree two in y. The Berwald connection coefficients G^i_j and G^i_{jk} can be derived from the function G^i, namely $G^i_j = G^i_{(j)}$, $G^i_{jk} = G^i_{j(k)}$. The Berwald covariant derivative can be written as

$$T^i_{j\shortparallel k} = \partial T^i_j/\partial x^k - T^i_{j(\alpha)}G^\alpha_k + T^\alpha_j G^i_{\alpha k} - T^i_\alpha G^\alpha_{jk}. \tag{1.1}$$

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler spaces on the underlying manifold M^n. A diffeomorphism $F^n \to \overline{F}^n$ is called *geodesic* if it maps an arbitrary geodesic curve of F^n to a geodesic curve of $\overline{F}^n$. In this case the change $L \to \overline{L}$ of the metrics is called *projective*. As it is well known ([3]), the mapping $F^n \to \overline{F}^n$ is *geodesic* (that is the change is projective) iff there exists a scalar field $p(x,y)$ satisfying

$$\overline{G}^i = G^i + py^i. \tag{1.2}$$

The projective factor $p(x,y)$ is a positively homogeneous function of degree 1 in y. From (1.2) we obtain

$$\overline{G}^i_j = G^i_j + p\delta^i_j + p_j y^i \quad ; \quad p_j = p_{(j)} \tag{1.3}$$

$$\overline{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p_{jk}y^i \quad ; \quad p_{jk} = p_{j(k)}. \tag{1.4}$$

A Finsler space F^n is said ([3]) to be of *scalar curvature* if

$$H^i_j = h^i_j H \tag{1.5}$$

where $h^i_j = \delta^i_j - l^i l_j$, $l^i = y^i/L$ and $l_i = L_{(i)}$. $H^i_j(x,y)$ is the deviation tensor of F^n which is given by ([3])

$$H^i_j = 2\partial G^i/\partial x^j - y^\alpha \partial G^i_j/\partial x^\alpha - 2G^i_{j\alpha}G^\alpha - G^i_\alpha G^\alpha_j, \quad \text{and} \quad H^\alpha_\alpha = (n-1)H. \tag{1.6}$$

From the curvature tensor $H^i_{hjk} = \frac{1}{3}(H^i_{k(j)} - H^i_{j(k)})_{(h)}$ we have $H^i_k = H^i_{\alpha\beta k}y^\alpha y^\beta$. (The index 0 denotes as usual the transvection by y^i, for instance $Q_\alpha y^\alpha = Q_0$.)

Assume that there exits a geodesic mapping between F^n and $\overline{F}^n$. Then the deviation tensors satisfy the relation

$$\overline{H}^i_j = H^i_j + y^i Q_{j\alpha} y^\alpha + \delta^i_j Q_\alpha y^\alpha - y^i Q_j \tag{1.7}$$

where $Q_j = p_{\shortparallel j} - pp_j$, $Q_{jk} = p_{j\shortparallel k} - p_{k\shortparallel j}$.

2. Recurrent Finsler spaces

Definition ([5]). A Finsler space F^n is called *recurrent* if the curvature tensor of F^n satisfies the following condition

$$H^i_{hjk\shortparallel l} = \lambda_l(x,y) H^i_{hjk} \tag{2.1}$$

where $\lambda_l(x,y)$ is a positively homogeneous function of degree 0 in y.

Several authors have studied the recurrent Finsler spaces and their generalisations (e.g. [4] , [6], [8], [9]).

Definition ([2]). A Finsler space F^n is called a *Landsberg space* if the condition

$$y_\alpha G^\alpha_{jkl} = -2P_{jkl} = 0 \tag{2.2}$$

holds good, where $g_{jk\shortparallel l} = -2P_{jkl}$ and $\partial G^i_{jk}/\partial y^l = G^i_{jkl}$.

Proposition 1. *For a recurrent Finsler space with the nonzero vector λ_l depends on the position x alone.*

Proof. Using the equation (2.1) from the integrabilty condition

$$T^i_{j\shortparallel k(l)} - T_{j(l)\shortparallel k} = T^\alpha_j G^i_{\alpha kl} - T^i_\alpha G^\alpha_{jkl}$$

we get

$$H^i_{jk}\lambda_{l(h)} = H^\alpha_{jk} G^i_{\alpha lh} + H^i_{\alpha j} G^\alpha_{klh} - H^i_{\alpha k} G^\alpha_{jlh} \tag{2.3}$$

where $H^i_{jk} = H^i_{\alpha jk} y^\alpha$. Contracting (2.3) by the y^k we have

$$H^i_j \lambda_{l(h)} = H^\alpha_j G^i_{\alpha lh} - H^i_\alpha G^\alpha_{jlh}. \tag{2.4}$$

After contracting (2.4) by the indices i and j we obtain $(n-1)H\lambda_{l(k)} =$ $= 0$. If we assume that $H \neq 0$, then $\lambda_{l(k)} = 0$ ◊

From Prop. 1 and from (2.4) follows

$$H^\alpha_j G^i_{\alpha lh} - H^i_\alpha G^\alpha_{jlh} = 0. \tag{2.5}$$

If F^n has scalar curvature, then substituting (1.5), (2.2) and (2.5) we obtain

$$\frac{H}{L} P_{jlh} = 0,$$

where $H = R(x,y)L^2$ and $R(x,y)$ is the curvature function. If $H \neq 0$ holds good (i.e. $R(x,y) \neq 0$) then F^n is a Lansberg space. From Numata's result [7], by which a Lansberg space of nonzero scalar curvature is a Riemannian space of nonzero constant curvature, follows the

Proposition 2. *A recurrent Finsler space F^n of nonzero scalar curvature is a Riemannian space of nonzero constant curvature.*

3. Geodesic mapping of recurrent Finsler spaces

In this section we suppose that $n > 2$. We are concerned with a geodesic mapping $F^n \to \overline{F}^n$, where F^n is an arbitrary but $\overline{F}^n$ is a recurrent Finsler space. In virtue of (1.1), (1.3) and (1.4)

$$(3.1)\quad \overline{H}^h_{i/\!/k} - (2p_k + \overline{\lambda}_k)\overline{H}^h_i - \overline{H}^h_{i(k)}p + \overline{H}^\alpha_i p_\alpha \delta^h_k + \overline{H}^\alpha_i p_{k\alpha} y^h - \overline{H}^h_k p_i = 0.$$

Here we use that $\overline{F}^n$ is a recurrent space (i.e. $\overline{H}^h_{ijl\overline{/\!/}k} = \overline{\lambda}_k \overline{H}^h_{ijl}$, from which $\overline{H}^h_{i\overline{/\!/}k} = \overline{\lambda}_k \overline{H}^h_i$), and the well-known identities ([2], [3]):

$$\overline{H}^h_{i(\alpha)} y^\alpha = 2\overline{H}^h_i \quad , \quad \overline{H}^h_\alpha y^\alpha = 0.$$

Substituting (1.7) into (3.1) and contracting it by the tensor h^l_h we have

$$(3.2)\quad \begin{aligned} &H^l_{i/\!/k} - h^l_i Q_{\alpha/\!/k} y^\alpha - (2p_k + \overline{\lambda}_k) H^l_i + (2p_k + \overline{\lambda}_k) h^l_i Q_\alpha y^\alpha - \\ &- H^l_k p_i - h^l_\alpha H^\alpha_{i(k)} p + + h^l_i (Q_\alpha y^\alpha)_{(k)} p + h^l_k H^\alpha_i p_\alpha = 0. \end{aligned}$$

Contracting (3.2) by y^k and indices l, i we receive

$$(3.3)\quad H_{/\!/\alpha} y^\alpha - Q_\alpha y^\alpha_{/\!/\beta} y^\beta - (4p - \overline{\lambda}) H + (2p + \overline{\lambda}) Q_\alpha y^\alpha + (Q_\alpha y^\alpha)_{(\beta)} y^\beta p = 0.$$

From (3.2) and (3.3) we can conclude that

$$(3.4)\quad A^l_{i/\!/\alpha} y^\alpha = (4p + \overline{\lambda}) A^l_i$$

where $A^l_i = H^l_i - H h^l_i$. (If $A^l_i = 0$, then F^n is a space of scalar curvature.) The *semi-recurrent* Finsler space is characterized by the equation (3.4) [1]. Therefore the following theorem is valid:

Theorem 1. *If a Finsler space F^n can be geodesically mapped onto a recurrent Finsler space $\overline{F}^n$, then F^n must be A-recurrent.*

Special cases

A) With help of the well-known identity $g_{i\alpha}H_j^\alpha - g_{j\alpha}H_i^\alpha = 0$ from (3.2) we obtain

$$\begin{aligned} &H_i^h h_j^\alpha p_\alpha - H_j^h h_i^\alpha p_\alpha - h_i^h H_j^\alpha p_\alpha + h_j^h H_i^\alpha p_\alpha + \\ &+2P_{j\alpha}^h H_i^\alpha - 2P_{i\alpha}^h H_j^\alpha + 2pC_{j\alpha}^h H_i^\alpha - 2pC_{i\alpha}^h H_j^\alpha = 0. \end{aligned} \tag{3.5}$$

Suppose that the F^n is a Riemannian space, then ([2])

$$\frac{1}{2}g_{ij(k)} = C_{ijk} = 0 \quad , \quad P_{ijk} = 0. \tag{3.6}$$

Contracting (3.5) by the indices h and i, then using (3.6) we get

$$H_\alpha^h p^\alpha = H h_\alpha^h p_\alpha \quad , \quad p^\alpha = g^{\alpha\beta} p_\beta,$$

i.e.

$$A_\alpha^i p^\alpha = 0. \tag{3.7}$$

Substituting (3.7), (3.6) and (3.5) we have

$$A_i^h h_j^\alpha p_\alpha - A_j^h h_i^\alpha p_\alpha = 0.$$

Using (3.7) this equation reduces to

$$A_i^h h_\beta^\alpha p_\alpha p^\beta = 0$$

which means that $A_i^h = 0$, if $h_\beta^\alpha p_\alpha p^\beta = p_\alpha p^\alpha - L^2 p^2 \neq 0$. Thus we have proved the following

Theorem 2. *If F^n is a Riemannian space which can be geodetically mapped onto a recurrent Finsler space $\overline{F}^n$, then F^n must be a Riemannian space of constant curvature if $p_\alpha p^\alpha - L^2 p^2 \neq 0$.*

B) Let F^n be also a recurrent Finsler space. From the equation

$$H_{ijk|l}^h = \lambda_l H_{ijk}^h$$

we obtain

$$H_{i|\alpha}^h y^\alpha = \lambda H_i^h \quad , \quad H_{|\alpha} y^\alpha = \lambda H \quad , \quad \lambda = \lambda_\alpha y^\alpha.$$

Thus

$$A_{i|\alpha}^h y^\alpha = \lambda A_i^h. \tag{3.8}$$

Considering (3.8) and (3.4) we get

$$(4p + \overline{\lambda} - \lambda) A_i^h = 0.$$

Finally we have the following

Theorem 3. *If a recurrent Finsler space F^n can be geodesically mapped onto a recurrent Finsler space $\overline{F}^n$, then F^n must be a Riemannian space of constant curvature if $4p + \overline{\lambda} - \lambda \neq 0$.*

The authors would like to express their gratitude to Professor M. Matsumoto for his valuable advice.

References

[1] BÁCSÓ, S.: A note on geodesic mapping on Finsler spaces, Colloquia Mathematica Societatis János Bolyai, 46, Topics in Differental Geometry, Debrecen, 1984, 131–136.

[2] MATSUMOTO, M.: Foundation of Finsler geometry and special Finsler spaces, Kaishcisha Press, Saikawa 3-23-2, Otsu-Shi, Shiga-Ken 520, Japan.

[3] MATSUMOTO, M.: Projective changes of Finsler metrics and projectively flat Finsler spaces, *Tensor* (N.S.) **34** (1980), 303-315.

[4] MIRSA, B. B.: On a recurrent Finsler space, *Rev. Roum. Math. Pure et Appl.* **18** (1973), 701–712.

[5] MOÓR, A.: Untersuchungen über Finslerräume von recurrenter Krünmung, *Tensor* (N.S.) **13** (1963), 1–18.

[6] MOÓR, A.: Unterräume von rekurrenter Krümmung in Finslerräumen, *Tensor* (N.S.) **24** (1972), 261-265.

[7] NUMATA, S.: On Lansberg spaces of scalar curvature, *J Korean Math. Soc.* **12** (1975), 97-100.

[8] PANDEY, P. N. and MIRSA, R. B.: Projective recurrent Finsler manifolds I., *Publ. Math. Debrecen* **28** (1981), 191–198.

[9] SEN, R. N.: Finsler spaces of recurrent curvature, *Tensor* (N.S.) **19** (1968), 291–299.