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ON (r, t)-COMMUTATIVITY OF $n_{(2)}$-PERMUTABLE SEMIGROUPS

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Abstract: In this paper we deal with the connection of $n_{(2)}$-permutable semigroups and (r,t)-commutative ones. We show that, for every pair of integers n and c with $n \geq 3$ and $n-1 \leq c \leq 2n-4$, there is a semigroup S such that $p(S) = n$ and $c(S) = c$, where $p(S)$ and $c(S)$ denote the degree of $n_{(2)}$-permutability and of commutativity of S, respectively. Moreover, we prove that, for every pair of integers n and m with $n \geq 5$ and $2 \leq m < 2n-4$, there is a semigroup which is $n_{(2)}$-permutable but not (r,t)-commutative for any r and t with $r+t=m$. By these results we have $2n-4 \leq \varphi(n) \leq 2n-3$ $(n \geq 5)$ which is a partial answer to the open problem raised in [5].

Throughout this paper $\mathbb{N}^+$ denotes the set of all positive integers. For notations and notions not defined here, we refer to [3]. As in [8], a semigroup S is said to be *$n_{(2)}$-permutable* ($n \in \mathbb{N}^+$, $n \geq 2$) if, for any n-tuple $(s_1, s_2, \ldots, s_n)$ of elements of S, there is an integer t with

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$1 \leq t \leq n-1$ such that

$$s_1 s_2 \cdots s_t s_{t+1} \cdots s_n = s_{t+1} \cdots s_n s_1 s_2 \cdots s_t.$$

By the *degree of $n_{(2)}$-permutability* of a semigroup S we shall mean an integer $p(S) \geq 2$ such that S is $p(S)_{(2)}$-permutable but not $(p(S)-1)_{(2)}$-permutable. Following [6], a semigroup S is said to be *(r,t)-commutative* ($r, t \in \mathbb{N}^+$) if for any elements $s_1, s_2, \ldots, s_{r+t}$ of S,

$$s_1 s_2 \cdots s_r s_{r+1} \cdots s_{r+t} = s_{r+1} \cdots s_{r+t} s_1 s_2 \cdots s_r.$$

In [1], it is proved that every (r,t)-commutative semigroup is also $(1, r+t)$-commutative. Thus we can define the *degree of commutativity* ([1]) of a semigroup S as an integer $c(S) \in \mathbb{N}^+$ such that S is $(1, c(S))$-commutative but not $(1, c(S)-1)$-commutative.

It is clear that an (r,t)-commutative semigroup is $(r+t)_{(2)}$-permutable. It is obvious that the converse is true if $r+t \leq 3$ (see [8]). By Lemma 2 of [8] and the fact that every (r,t)-commutative semigroup is (r', t')-commutative for every $r' \geq r$, $t' \geq t$, it follows that, for every integer $n \geq 4$, there is a semigroup which is $n_{(2)}$-permutable but not (r,t)-commutative for any positive integers r and t with $r+t \leq n$. In [8], the author raised the following question. Does $n_{(2)}$-permutability ($n \geq 4$) of a semigroup S imply (r,t)-commutativity of S for some r and t. In [2], the authors gave a positive answer to this question for finite semigroups. In [4], M. Gutan proved that the answer is positive for arbitrary semigroups. He proved that every $n_{(2)}$-permutable semigroup is $(n-1, n-1)$-commutative ($n \geq 4$). In [5], he also proved that every $n_{(2)}$-permutable semigroup is $(1, 2n-4)$-commutative ($n \geq 4$). Denoting by $\mathcal{P}_{m,n}(m, n \in \mathbb{N}^+,\ n \geq 2)$ the proposition

$\mathcal{P}_{m,n}$: If S is an arbitrary $n_{(2)}$-permutable semigroup then there exist r and t in $\mathbb{N}^+$ with $r+t=m$ such that S is (r,t)-commutative,

consider

$$\varphi(n) = \min\{m \in \mathbb{N}^+; \mathcal{P}_{m,n} \text{ is true}\}.$$

It is evident that $\varphi(2) = 2$ and, by [8], $\varphi(3) = 3$. In [5], it was proved that $\varphi(n) \leq 2n-3$ and so $\varphi(4) = 5$. It is an open problem to find $\varphi(n)$ for $n \geq 5$ (see [5]). It is obvious that $\varphi(n) \geq n$ (see also Lemma 2 of [8]). In this paper we show that $2n-4 \leq \varphi(n) \leq 2n-3$ for $n \geq 5$.

For a product $s_1 s_2 \cdots s_n$ of elements s_i ($i = 1, 2, \ldots, n$) of a semigroup S let $p_i = s_i \cdots s_n s_1 \cdots s_{i-1}$ and $I_{p_i} = \{j \in \{1, 2, \ldots, n\};\ p_i = p_j\}$. We note that s_0 denotes the identity element of S^1. The following lemma plays an important role in our investigations.

Lemma 1. *If S is an $n_{(2)}$-permutable semigroup then, for every nonnegative integer k and $p_1 = s_1 s_2 \cdots s_{n+k} \in S^{n+k}$, the cardinality of I_{p_1} is at least $k+2$.*

Proof. We proceed by induction on k. Let $|I_{p_1}|$ denote the cardinality of I_{p_1}. If $k = 0$ then $|I_{p_1}| \geq 2$ for every $p_1 \in S^n$, because S is $n_{(2)}$-permutable. Assume that $|I_{p_1}| \geq k+2$ for some nonnegative integer k and for every $p_1 \in S^{n+k}$. Let $s_1, s_2, \ldots, s_{n+k+1}$ be arbitrary elements of S. As S is an $n_{(2)}$-permutable semigroup, by Lemma 1 of [8], S is also $(n+k+1)_{(2)}$-permutable. Hence, there is an index $i \in \{2, \ldots, n+k+1\}$ such that $p_1 = p_i$. Consider the product $q = s_1 s_2 \cdots (s_{i-1} s_i) \cdots s_{n+k+1} \in S^{n+k}$. By the assumption $|I_q| \geq k+2$. As $|I_q| < |I_{p_1}|$, therefore $|I_{p_1}| \geq k+3$. ◊

Construction. Let $\mathcal{F}_X$ be the free semigroup (without the empty word) over the set $X = \{a, b\}$. If $w \in \mathcal{F}_X$ then $l(w)$ denotes the length of w. Let z be an integer with $z \geq 4$. Consider the pairwise disjoint subsets A_z, B_z C_z, D_z of $\mathcal{F}_X$ defined as follows. Let

$$A_z = \{a^z\},$$

$$B_z = \left\{ a^{z-(2g-1)} b a^{2g-2}; \quad g = 1, 2, \ldots \left[\frac{z+1}{2}\right] \right\},$$

$$C_z = \left\{ a^{z-2h} b a^{2h-1}; \quad h = 1, 2, \ldots \left[\frac{z}{2}\right] \right\},$$

$$D_z = \{w \in \mathcal{F}_X; \quad l(w) = z\} - (A_z \cup B_z \cup C_z).$$

For a fixed integer $n \geq 4$ and for an arbitrary nonnegative integer k, define the following congruence

$$\alpha_{n+k} = \{(w_1, w_2) \in \mathcal{F}_X \times \mathcal{F}_X : \ w_1 = w_2 \text{ or } l(w_1), l(w_2) > n+k \text{ or}$$

$$\exists\ 0 \leq i, j, t, \leq k \ (w_1, w_2 \in B_{n+i} \text{ or } w_1, w_2 \in C_{n+j} \text{ or } w_1, w_2 \in D_{n+t})\}.$$

Let $S_{n+k} = \mathcal{F}_X / \alpha_{n+k}$.

Theorem 1. *The factor semigroup S_{n+k} is $n_{(2)}$-permutable if and only if $k \leq n-4$. In this case $p(S_{n+k}) = n$.*

Proof. Assume that S_{n+k} is $n_{(2)}$-permutable. As the length of the elements of B_{n+k} and C_{n+k} is $n+k$, both of B_{n+k} and C_{n+k} have at least $k+2$ elements (see Lemma 1). Hence $|B_{n+k} \cup C_{n+k}| \geq 2k+4$. On the other hand $|B_{n+k} \cup C_{n+k}| = n+k$. Therefore, $2k+4 \leq n+k$ from which we get $k \leq n-4$.

Conversely, assume that $k \leq n-4$. Let $s_1, s_2, \ldots, s_n \in S_{n+k}$ be arbitrary elements. Consider words $q_i \in \mathcal{F}_X$ such that $q_i \alpha_{n+k} = s_i$

$(i=1,2,\ldots,n)$. If $l(q_1q_2\cdots q_n)>n+k$ then $(q_1q_2\cdots q_n, q_2\cdots q_nq_1)\in \in \alpha_{n+k}$ and so $s_1s_2\cdots s_n = s_2\cdots s_ns_1$. Assume $l(q_1q_2\cdots q_n)\leq n+k$. Then there is an integer $i\in\{0,1,\ldots,k\}$ such that $l(q_1q_2\cdots q_n) = = n+i$. If $q_1q_2\cdots q_n \in D_{n+i}$ then $(q_1q_2\cdots q_n,\ q_2\cdots q_nq_1)\in\alpha_{n+k}$ and so $s_1s_2\cdots s_n = s_2\cdots s_ns_1$. Assume $q_1q_2\cdots q_n\in B_{n+i}$. Then there is an index $j\in\{1,2,\ldots,n\}$ such that the word q_j contains the letter b as a factor (and so $q_1,q_2,\ldots,q_{j-1},q_{j+1},\ldots q_n$ do not contain b). Assume that $l(q_1q_2\cdots q_t)$ and $l(q_r\cdots q_n)$ are odd numbers for all $t\in\{1,2,\ldots j-1\}$ and $r\in\{j+1,\ldots,n\}$. If $j=1$ then $l(q_n)$ is odd and $l(q_r)$ $r=2,3,\ldots,n-1$ is even. Hence

$$l(q_1q_2\cdots q_n)\geq 2(n-2)+2>2n-4.$$

This is a contradiction. In case $j=n$ we can get a contradiction in a similar way. Assume $j\notin\{1,n\}$. Then $l(q_1)$ and $l(q_n)$ are odd and $l(q_r)$ is even for every $r=2,3,\ldots j-1,j+1,\ldots,n-1$. Therefore,

$$l(q_1q_2\cdots q_n)\geq 2(n-3)+3>2n-4$$

which is a contradiction. Consequently, $l(q_1q_2\cdots q_t)$ or $l(q_r\cdots q_n)$ is even for some $t\in\{1,2,\ldots j-1\}$ and $r\in\{j+1,\ldots,n\}$. Thus $q_{t+1}\cdots q_nq_1\cdots q_t\in B_{n+i}$ or $q_r\cdots q_nq_1\cdots q_{r-1}\in B_{n+i}$ and so

$$s_1s_2\cdots s_n = s_{t+1}\cdots s_ns_1s_2\cdots s_t \text{ or } s_1s_2\cdots s_n = s_r\cdots s_ns_1s_2\cdots s_{r-1}$$

for some $t\in\{1,2,\ldots j-1\}$ and $r\in\{j+1,\ldots,n\}$. We get a similar result in case $q_1q_2\ldots q_n\in C_{n+i}$. If $q_1q_2\cdots q_n\in A_{n+i}$ then $s_1s_2\cdots s_n = = s_2\cdots s_ns_1$. Thus S_{n+k} is $n_{(2)}$-permutable. As S_{n+k} is not $(n-1)_{(2)}$-permutable, $p(S_{n+k})=n$. $\Diamond$

Corollary 1. *The semigroup S_{n+k} $(4\leq n,\ 0\leq k\leq n-4)$ is $(1,n+k)$-commutative, but not $(1,n+k-1)$-commutative.*

Proof. It is clear that S_{n+k} is $(1,n+k)$-commutative. As $B_{n+k}\cap \cap C_{n+k}=\emptyset$, the semigroup S_{n+k} is not $(1,n+k-1)$-commutative. $\Diamond$

Next we deal with the connection between $p(S)$ and $c(S)$ for an arbitrary semigroup.

Theorem 2. *For every integers n and c with $n\geq 3$ and $n-1\leq c\leq \leq 2n-4$, there is a semigroup S such that $p(S)=n$ and $c(S)=c$.*

Proof. By Remark 2 of [8], a semigroup is $3_{(2)}$-permutable if and only if it is $(1,2)$-commutative. Assume $n\geq 4$. It is evident that every $(1,t)$-commutative semigroup is $(t+1)_{(2)}$-permutable. From this it follows that $c(S)<p(S)$ implies $c(S)=p(S)-1$. By the construction contained in the proof of Th. 3 of [1] or Th. 3 of [9], it follows that there are semigroups S such that $p(S)=n$ and $c(S)=n-1$. Therefore, we

can suppose that $n \leq c \leq 2n-4$. Then, for the semigroup S_c defined above, $p(S) = n$ and $c(S) = c$. ◊

Theorem 3. *For every pair of integers n and m with $n \geq 5$ and $2 \leq \leq m < 2n-4$, there is a semigroup which is $n_{(2)}$-permutable but not (r,t)-commutative for any r and t with $r+t = m$.*

Proof. Let n be an arbitrary integer with $n \geq 5$. If m is an integer with $2 \leq m \leq n$ then the assertion is true (see Lemma 2 of [8]). Assume that $n < m < 2n-4$ for some integer m. Then $n \geq 6$. Let k be a positive integer such that $m = n+k-1$. Clearly, $2 \leq k \leq n- -4$. Consider the semigroup S_{n+k} defined in the Construction. By Th. 1, S_{n+k} is $n_{(2)}$-permutable. Assume that S_{n+k} is (r,t)-commutative for some positive integers r and t with $r+t = m = n+k-1$. If r is odd then consider the element $a^{n+k-2}b \in B_{n+k-1}$. Since S_{n+k} is (r,t)-commutative, we obtain that $a^{n+k-2}b$, $a^{n+k-r-2}ba^r \in \alpha_{n+k}$ and so $a^{n+k-r-2}ba^r \in B_{n+k-1}$. However, since r is odd this is impossible. If r is even then $a^{n+k-1}b$, $a^{n+k-r-2}ba^2a^{r-1} \in B_{n+k}$ and so the parity of $n+k-1$ and $n+k-r$ must be the same. This is also impossible. Consequently S_{n+k} is not (r,t)-commutative for any r and t with $r+t = m$. ◊

Corollary 2. *For every integer $n \geq 5$, we have $2n-4 \leq \varphi(n) \leq 2n-3$.*

Proof. By Th. 3, if $\mathcal{P}_{m,n}$ is true for some positive integers m and n with $n \geq 5$ then $m \geq 2n-4$. Thus $\varphi(n) \geq 2n-4$. This and the fact $\varphi(n) \leq 2n-3$ proved in [5] together imply our assertion. ◊

The following lemma is an addendum to the problem of finding the exact value of $\varphi(n)$.

Lemma 2. *If an $n_{(2)}$-permutable semigroup S ($n \geq 4$) is (r,t)-commutative for some r and t with $r+t = 2n-4$ then either r and t are even or S is $(1, 2n-5)$-commutative.*

Proof. Assume that S is a semigroup such that it is $n_{(2)}$-permutable and (r,t)-commutative for some integers n, r and t with $n \geq 4$, $r+t = 2n-4$. Assume that S is not $(1, 2n-5)$-commutative. Let d denote the greatest common divisor of t and r. By Cor. 1 of [1], S is $(hd, 2n-4-hd)$-commutative for every $h = 1, 2, \ldots, \frac{2n-4}{d} - 1$. Then $d \geq 2$. We can suppose that $d > 2$. As S is not $(1, 2n-5)$-commutative, there are elements $s_1, s_2, \ldots s_{2n-4}$ of S such that

$$p_1 = s_1 s_2 \cdots s_{2n-4} \neq s_2 \cdots s_{2n-4} s_1 = p_2.$$

By Lemma 1, $|I_{p_1}|, |I_{p_2}| \geq n-2$. As $I_{p_1} \cap I_{p_2} = \emptyset$, $|I_{p_1}| = |I_{p_2}| = n-2$.

For $i = 0, 1, \dots d-1$, let

$$J_i = \left\{(h-1)d + i + 1;\ h = 1, 2, \dots, \frac{2n-4}{d}\right\}.$$

It is easy to see that J_i contained in either I_{p_1} or I_{p_2} for every $i = 0, 1, \dots d-1$. Moreover

$$\bigcup_{i=0}^{d-1} J_i = \{1, 2, \dots, 2n-4\}.$$

Therefore, $n-2 = \frac{2(n-2)}{d}g$ for some positive integer g. From this it follows that $d = 2g$. Thus r and t are even.◊

We note that, from Lemma 2, it follows that if a semigroup S is $n_{(2)}$-permutable and (r,t)-commutative such that $n-2$ is a prime, $n \geq 4$ and $r + t = 2n-4$ then S is $(2, 2n-6)$-commutative.

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