Mathematica Pannonica
6/1 (1995), 105 – 114

CONNECTIONS IN K-VECTOR BUNDLES

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Dedicated to Prof. Dr. Hans Vogler on the occasion of his 60th birthday

Received January 1995

MSC 1991: 53 B 40, 53 C 60

Keywords: Areal space, Grassmann cone bundle, connection

Abstract: In the paper the affine theory of areal spaces is investigated. There is given a relationship between the positively homogeneous connections of the Grassmann cone bundle $Z^k\tau_M$ and those of the Whitney sum $\overset{k}{\bigoplus}\tau_M$ of order k. This method leads to a characterization of Riemannian metrizability of linear connections on a manifold.

1. Introduction: Area and areal spaces

In the usual differential geometric spaces the basic metrical notion is the arclength and the area of different dimensional submanifolds is a deduced concept only. (Differently from this, in an areal space [5, 6] the starting point is the area.) This is well known in Riemannian spaces V_n. In a Finsler space $F_n = (M, L)$ with an n dimensional base manifold and fundamental function L the area can be deduced in the following way [10, 13]. Let x^i, $i = 1, \dots, n$ be local coordinates on

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This research was supported by OTKA 1653 and F4504.

$U \subset M$ and y^i in the tangent space $T_{x_0}M$. Then $L(x_0, y) = 1$ is the indicatrix $I(x_0)$. $I(x)$, playing the role of the unit sphere, determines the norms of the vectors of T_xM, and thus makes T_xM into a finite dimensional Banach space, i.e. into a Minkowski space M_n. The solid body determined by I will be denoted by $\mathcal{B}$. Let $\Phi \subset M$ be a k-dimensional $(k < n)$ submanifold of M given in local coordinates by $x^i = x^i(u^1, \dots, u^n)$ (u is taken from a parameter domain B) and $p(u)$ an infinitesimal surface element of Φ laying in the tangent space $\sigma \subset \subset T_{x(u)}M$. Since Minkowski areas of domains D_1 and D_2 of the same dimension and laying in the same linear subspace of T_xM are related as their euclidean areas: $\|D_1\|_M : \|D_2\|_M = \|D_1\|_E : \|D_2\|_E$, where $\| \, . \, \|_E$ is the euclidean area-measure in an arbitrary euclidean metric of T_xM [3], we have

$$\text{(1)} \qquad \|p(u)\|_M : \|\mathcal{B} \cap \sigma\|_M = \|p(u)\|_E : \|\mathcal{B} \cap \sigma\|_E.$$

Since $\mathcal{B}$ plays the role of the solid unit sphere (n-dimensional disk), it is quite natural to define $\|\mathcal{B} \cap \sigma\|$ as the value of the area of the k-dimensional sphere which will be denoted by $\omega^{(k)}$. Thus (1) can be considered as the definition of $\|p(u)\|_M$, for in (1) everything is defined except $\|p(u)\|_M$. Furthermore an F_n is infinitesimally an M_n. Hence the Finsler measure $\|p(u)\|_F$ of the infinitesimal $p(u)$ equals $\|p(u)\|_M$. Thus one can define for the Finsler measure of Φ

$$\text{(2)} \qquad \|\Phi\|_F = \int_B \|p(u)\|_F du.$$

Other recent investigations touching upon Finsler area can be found in [1]. $\|p(u)\|_F$ depends on $I(x(u))$ and thus indirectly on the given fundamental function L. Hence the integrand in (2) can be expressed as $\bar{F}(x, p)$:

$$\|\Phi\|_F = \int_B \bar{F}(x(u), p(u)) du.$$

Finally it is clear that the function $\bar{F}$ must be positively homogeneous of order one in p. We mean by λp a k-dimensional domain in σ such that $\|\lambda p\|_E = |\lambda| \cdot \|p\|_E$. Then as a consequence of (1) we obtain $\bar{F}(x, \lambda p) = |\lambda| \bar{F}(x, p)$.

An *areal space* $\mathcal{A}_n^{(k)}$ [5, 6] is locally a couple (M, F) of a manifold and a function

$$F: U \times GK_{n,k} \to \mathbb{R}^+, \quad (x, p) \mapsto F(x, p), \quad U \subset M$$

positively homogeneous of order one in p, where $GK_{n,k}$ means the Grassmann cone [2, 9], whose elements can be represented as parallelotops $\mathcal{P}$ spanned by k vectors $v^1, \dots, v^k \in \sigma \subset \mathbb{R}^n$, where σ is a k-dimensional linear subspace of $\mathbb{R}^n$. p can be expressed in form of a k-vector as $p = v^1 \wedge \dots \wedge v^k$. $\{\sigma\}$ is the total space of the Grassmann manifold $G_{n,k}$ [14, 2]. – Then the *areal measure* $\|\Phi\|_A$ of Φ is defined by [5, 6, 12]

$$\|\Phi\|_A = \int_B F(x(u), p(u))\, du,$$

where $p = \frac{\partial x}{\partial u^1} \wedge \dots \wedge \frac{\partial x}{\partial u^k}$. We remark that $F(x,p)$, and thus the area measure of an $\mathcal{A}_n^{(k)}$ cannot be deduced in general from a Finsler space F_n ([11]). This means that areal spaces are more general than Finsler spaces with respect to area measuring.

2. Connections in $\mathcal{A}_n^{(k)}$ and in vector bundles

1. The role of the vectors of a Riemannian or Finsler geometry is taken over in an $\mathcal{A}_n^{(k)}$ by the elements of $GK_{n,k}$. Thus connections of areal spaces can and must be defined in the fibre bundle, called the Grassmann cone bundle of M

$$Z^k\tau_M = (Z^kTM, \bar{\pi}, GK_{n,k}, M)$$

where Z^kTM is the total space, $\bar{\pi}$ is the projection operator, $GK_{n,k}$ is the typical fibre and M is the base manifold.

Unfortunetaly, $GK_{n,k}$ is no vector space and this can make much incovenience. We want to show that a homogeneous nonlinear connection $H_{Z^k\tau_M}$ of $Z^k\tau_M$ [4, 8] can be identified with a certain special homogeneous nonlinear connection $H_{\overset{k}{\oplus}\tau_M}$ on a vector bundle, the k-Whitney sum of the tangent bundle τ_M

$$\overset{k}{\oplus}\tau_M = (\overset{k}{\oplus}TM, \pi, \mathbb{R}^{kn}, M)$$

with a $k{\cdot}n$ dimensional vector space as fibre over the same base manifold M as $Z^k\tau_M$, and conversely, every such special $H_{\overset{k}{\oplus}\tau_M}$ determines a $H_{Z^k\tau_M}$: $H_{Z^k\tau_M} \Longleftrightarrow \text{spec } H_{\overset{k}{\oplus}\tau_M}$.

A nonlinear connection $H_{\overset{k}{\oplus}\tau_M}$ in $\overset{k}{\oplus}\tau_M$ is given by the splitting

$T_zE = V_zE \oplus H_zE$, $\overset{k}{\oplus} TM = E$, $z \in E$. Let x^i be local coordinates in $U \subset M$, then (x^i, y^a), $i = 1, \dots, n$, $a = 1, \dots, kn$ are local coordinates of $z \in \pi^{-1}(U) \subset E$. Then H_zE is spanned by

$$\delta_i = \frac{\partial}{\partial x^i} - N_i^a(x, y) \frac{\partial}{\partial y^a}$$

where $N_i^a(x, y)$ are the connection coefficients. Positive homogeneity means that for $\mu_t: (x^i, y^a) \mapsto (x^i, ty^a)$

$$(d\mu_t)\delta_i(x, y) = \delta_i(x, ty)$$

which is equivalent with $N_i^a(x, ty) = tN_i^a(x, y)$.

2. Now let y be an element of $\pi^{-1}(x) \cong \mathbb{R}^{kn}$ with the components y^a, $a = 1, \dots, kn$ and let $\overset{\alpha}{v}$ $\alpha = 1, \dots, k$ be k vectors in $\mathbb{R}^n$ with components $\overset{\alpha i}{v} = y^{(\alpha - 1)n + i}$. Thus $y \leftrightarrow (\overset{1}{v}, \dots, \overset{k}{v}) = \mathcal{P}$ which $\mathcal{P}$ is a representation of a $p \in GK_{n,k}$. This representation can be considered as a mapping $\varrho: GK_{n,k} \to \{\mathcal{P}\}$. ϱ is multivalent and onto, while ϱ^{-1} is univalent. $\mathcal{P}_1$ and $\mathcal{P}_2$ are two representations of p, in signs: $\mathcal{P}_1 \sim \mathcal{P}_2$, iff A) they lay in the same k-dimensional linear subspace σ, and B) they have the same volume with the same sign. $\sim$ is an equivalence relation. Thus each p can be identified wits its equivalence class. The other elements of the equivalence class of a $y_0 = (\overset{1}{v}_0, \dots, \overset{k}{v}_0)$ are, because of A), $y = (\overset{1}{v}, \dots, \overset{k}{v})$ with

$$\overset{\alpha}{v} = t_\beta^\alpha \overset{\beta}{v}_0, \quad \alpha, \beta = 1, 2, \dots, k \tag{3}$$

and then, because of B)

$$\mathrm{Det}|t_\beta^\alpha| = +1$$

Conversely, such transformations take an element of a class always into another element of the class. This means that any equivalence class is generated from one of its elements by special unimodular linear transformations sl. Their set is denoted by $\mathcal{S}l$, and by matrix multipilication $\mathcal{S}l$ becomes a group. The whole class of y_0 is $\mathcal{S}l\, y_0$.

3. Our idea is the following. Given a linear connection $H_{\overset{k}{\oplus} TM}$, it takes $y_0 \in \pi^{-1}(x)$ into a $\tilde{y}_0 \in \pi^{-1}(x + dx)$ and a $y \in \mathcal{S}l\, y_0$ into a $\tilde{y}$. If these $\tilde{y}$ form also an equivalence class in $\pi^{-1}(x + dx)$, i.e. if also $\tilde{y} \in \mathcal{S}l\, \tilde{y}_0$ holds, then we obtain a $p(x) \to \tilde{p}(x + dx)$ and this determines a $H_{Z^k TM}$. We want to obtain conditions for $\tilde{y} \in \mathcal{S}l\, \tilde{y}_0$.

Let $\gamma := x(t)$ be a curve in M, $z_0 = (x_0, y_0)$, $x_0 = x(t_0)$ an element of $\pi^{-1}(x_0)$ and $\tilde{z}_0(t) = (x(t), \tilde{y}_0(t))$ the parallel transport of z_0 along γ. If x_0 and $x_0 + dx$ are neighbouring points on γ, then $\tilde{y}^a - y_0^a = N_i^a(x_0, y_0)dx^i + o(dx)$ ($o(dx)$ means the terms of order higher than 1 in dx), i.e.

$$dy^a = N_i^a(x_0, y_0)dx^i \tag{4}$$

Conversely, if (5) holds everywhere along γ, then $\tilde{y}_0(t)$ is the parallel translated vector of y_0.

Let be $y_0 = (\overset{1}{v}_0, \dots, \overset{k}{v}_0)$ and $y = (\overset{1}{v}, \dots, \overset{k}{v}) \in \mathcal{S}l\, y_0$, moreover let $\tilde{y}_0 = (\overset{1}{\tilde{v}}_0, \dots, \overset{k}{\tilde{v}}_0)$ and $\tilde{y} = (\overset{1}{\tilde{v}}, \dots, \overset{k}{\tilde{v}})$ be their parallel translated from x_0 to $x_0 + dx$. Then in components

$$\overset{\alpha j}{\tilde{v}}_0 = \overset{\alpha j}{v}_0 + N_i^{(\alpha-1)n+j}(x_0, y_0)\, dx^i + o(dx) \tag{5}$$

$$\overset{\alpha j}{\tilde{v}} = \overset{\alpha j}{v} + N_i^{(\alpha-1)n+j}(x_0, y)\, dx^i + o(dx), \tag{6}$$

where $\overset{\alpha j}{\tilde{v}}_0$ is the j-th component of the vector $\overset{\alpha}{\tilde{v}}_0$.

Since $\tilde{y} \in \mathcal{S}l\, \tilde{y}_0$, we obtain $\overset{\alpha}{\tilde{v}} = s_\beta^\alpha \overset{\beta}{\tilde{v}}_0$ with $\mathrm{Det}|t_\beta^\alpha| = 1$. s_β^α depends on x_0 and dx. Therefore

$$s_\beta^\alpha(x_0 + dx) = s_\beta^\alpha(x_0) + s_{\beta i}^\alpha\, dx^i + o\,(dx)$$

where $s_{\beta i}^\alpha$ are the partial derivatives of s_β^α at x_0. In components

$$\begin{aligned} \overset{\alpha j}{\tilde{v}} &= \overset{\alpha j}{v} + N_i^{(\alpha-1)n+j}(x_0, y)\, dx^i + o(dx) = \\ &= s_\beta^\alpha(x_0 + dx)\left(\overset{\beta j}{v}_0 + N_i^{(\beta-1)n+j}(x_0, y_0)\, dx^i + o(dx)\right) = \\ &= s_\beta^\alpha(x_0 + dx)\overset{\beta j}{\tilde{v}}_0 \end{aligned} \tag{7}$$

and hence

$$\begin{aligned} &(\overset{\alpha j}{v} - (s_\beta^\alpha(x_0) + s_{\beta i}^\alpha\, dx^i)\overset{\beta j}{v}_0) + (N_i^{(\alpha-1)n+j}(x_0, y_0) - \\ &-(s_\beta^\alpha(x_0) + s_{\beta i}^\alpha\, dx^i)N_i^{(\beta-1)n+j}(x_0, y_0))\, dx^i + o(dx) = 0. \end{aligned} \tag{8}$$

These must hold on any curve starting from x, i.e. (6)–(8) must hold for any dx. Then (8) yields $\overset{\alpha j}{v} = s_\beta^\alpha \overset{\beta j}{v}_0$. Comparing this with (3) we obtain

$$s_\beta^\alpha = t_\beta^\alpha. \tag{9}$$

In view of this we get from (8) that

$$(10)\quad N_i^{(\alpha-1)n+j}(x,y) = t^\alpha_\beta N_i^{(\beta-1)n+j}(x,y_0) + s^\alpha_{\beta i}\overset{\beta}{v}{}^j_0 \quad \forall\, x, \forall\, \overset{\alpha}{v} = t^\alpha_\beta \overset{\beta}{v}_0$$

We obtained that if $H_{\overset{k}{\oplus}\tau_M}$ preserves equivalence classes then N_i^α must satisfy (10) with $\mathrm{Det}|t^\alpha_\beta| = +1$ and with some $(s^\alpha_{\beta i})$.

We show that (10) also suffices for this. First we recall that $\overset{\alpha}{\tilde{v}}_0$ and $\overset{\alpha}{\tilde{v}}$ are parallel translated vectors of $\overset{\alpha}{v}_0$ and $\overset{\alpha}{v}$ resp. iff (6) holds up to terms linear in dx (up to $o(dx)$). Therefore, concerning parallelity the last terms in (5–8) are unimportant if these equations hold otherwise for any x and dx. Now from (3), (9) and (10) follow (7) and (5) up to linear terms in dx, and this means that $\tilde{y} \in \mathcal{S}l\tilde{y}_0$. Thus we have obtained

Proposition. $H_{\overset{k}{\oplus}\tau_M}$ *preserves equivalence classes defined by* $\mathcal{S}l$ *iff its connection coefficients* $N^a(x,y)$ *satisfy* (10).

Such $H_{\overset{k}{\oplus}\tau_M}$ are called special and will be denoted by sp $H_{\overset{k}{\oplus}\tau_M}$. We show that any sp $H_{\overset{k}{\oplus}\tau_M}$ determines a homogeneous $H_{Z^k\tau_M}$, and conversely.

$p \in GK_{n,k}$ can be considered as a simple p-vector (see [9]) $p = \overset{1}{v}_0 \wedge \ldots \wedge \overset{k}{v}_0 = \overset{1}{v} \wedge \ldots \wedge \overset{k}{v}$. Then $p^{j_1\cdots j_k}$ is the value of the $k \times k$ determinant formed from the j_1-th, ..., j_k-th columns of the $n \times k$ matrix $(\overset{\alpha}{v}{}^j)$. These $p^{j_1\cdots j_k}$ are the components of the simple k-vector p and represent local coordinates for p over a neighborhood $U_p \subset GK_{n,k}$. $\tilde{p} = \overset{1}{\tilde{v}}_0 \wedge \ldots \wedge \overset{k}{\tilde{v}}_0 = \overset{1}{\tilde{v}} \wedge \ldots \wedge \overset{k}{\tilde{v}} = (\overset{1}{v} + d\overset{1}{v}) \wedge \ldots \wedge (\overset{k}{v} + d\overset{k}{v}) = p + dp(x,dx) + o(dx)$, where dp is linear in dx. $H_{Z^k\tau_M}$ is defined by these dp which depend on $d\overset{\alpha}{v}$, and so on the given sp $H_{\overset{k}{\oplus}\tau_M}$. We also show that $H_{Z^k\tau_M}$ determined by a sp $H_{\overset{k}{\oplus}\tau_M}$ is a homogeneous connection. This is true if $H_{Z^k\tau_M}$ takes tp into $t\tilde{p}$ supposed that it takes p into $\tilde{p}$. We know that the representation ϱ is homogeneous, i.e. ty_0 is a representation of tp: $\varrho^{-1}(ty_0) = tp$ provided $\varrho^{-1}(y_0) = p$. Then $H_{Z^k\tau_M}$ takes tp into $\varrho^{-1}(\widetilde{ty_0})$ which is $\varrho^{-1}(t\tilde{y}_0)$ because of the homogeneity of ϱ.

Conversely, also a positively homogeneous connection $H_{Z^k\tau_M}$ in $Z^k\tau_M$ determines a special connection in $\overset{k}{\oplus}\tau_M$. Really, given a curve $\gamma : x(t) \subset M$ the $H_{Z^k\tau_M}$ determines the parallel translated $p(x(t)) =$

$= p(t)$ of a $p(t_0) \in \bar{\pi}^{-1}(x(t_0))$. Then $p(t) = \overset{1}{v}(t) \wedge \ldots \wedge \overset{k}{v}(t)$, hence $\varrho(p(t)) = z(t) = (\overset{1}{v}(t), \ldots, \overset{k}{v}(t))$ means curves in $\overset{k}{\oplus} TM$ through each $z_0 = (\overset{1}{v}(0), \ldots, \overset{k}{v}(0)) \in \varrho(p(t_0))$. These $z(t)$ form an equivalence class in $\pi^{-1}(x(t))$. Considering n curves γ_i, $i = 1, \ldots, n$ having linearly independent tangents $\dot{\gamma}_i(t_0)$, we get through any z_0 n curves $z_i(t)$. Their tangents are never vertical and span up a linear n-dimensional subspace in $T_{z_0}(\overset{k}{\oplus} TM)$ which can be considered as horizontal subspace of $T_{z_0}E$. Performing this for each x and z we obtain a positively homogeneous connection in $\overset{k}{\oplus}\tau_M$. This connection takes equivalence classes into equivalence classes. Hence it is a special connection sp $H_{\overset{k}{\oplus}\tau_M}$, and thus its connection coefficients satisfy (9). Moreover this special connection of $\overset{k}{\oplus}\tau_M$ induces the given connection of $Z^k\tau_M$.

Theorem 1. *A positively homogeneous connection $H_{Z^k\tau_M}$ is equivalent in the considered representation of $GK_{n,k}$ with a special vector bundle connection $H_{\overset{k}{\oplus}\tau_M}$ which satisfies* (10) *and so preserves certain equivalence classes.*

3. Riemannian metrizability of symmetrical linear connections

We apply the ideas of the previous section in order to investigate the Riemann-metrizability of a linear connnection Γ without torsion.

A torsion free linear connection Γ on a manifold M is called *metrizable* if there is at least one covariant constant, symmetrical, positive definite 2-form g; in local coordinates: there exists a $g_{ij}(x)$ such that $g_{ij}(x) = g_{ji}(x)$, $g_{ij}(x)\xi^i\xi^j > 0$ $\forall \xi \neq 0$, and $\nabla_k g_{ij} = 0$. This is equivalent with the existence of a field of ellipsoids $g_{ij}(x)\xi^i\xi^j = 1$ in the tangent spaces denoted by $I(x)$ and called indicatrices which are absolute parallel, i.e. the parallel translation of an $I(x_0)$ along any curve to x yields $I(x)$.

$I(x)$ is determined by n conjugate diameters which can be replaced by n linearly independent vectors $\overset{\alpha}{v}$ $\alpha = 1, \ldots, n$ showing from the origin to an endpoint of a diameter. The parallel displaced $\overset{\alpha}{v}(x, \gamma)$ of these $\overset{\alpha}{v}$ from x_0 to an arbitrary x along a curve γ form in general, even in the case of a metrical Γ, no absolute parallel vector fields, but they always form conjugate axes of ellipsoids depending on γ and x. However in a metrical connection Γ these ellipsoids are the same at one point, they depend on the point x and are independent of the curve γ; and conversely, if the ellipsoids determined by the parallel displaced of $\overset{\alpha}{v}$ depend on x alone, then Γ is metrizable, and the $g_{ik}(x)$ sought for are the coefficents in the equation $g_{ik}(x)\xi^i\xi^k = 1$.

Given Γ, in a local coordinate system by $\Gamma^i_{jk}(x)$, consider the vector bundle $\overset{n}{\oplus}\tau_M = (\overset{n}{\oplus}TM, \pi, \mathbb{R}^{n\cdot n}, M)$ and the connection $H_{\overset{n}{\oplus}\tau_M}$ with local coefficients

$$N_i^a(x,y) := \Gamma_p{}^r{}_i(x)\overset{k}{v}{}^p \quad a = 1, \dots, n^2 \tag{11}$$
$$a = (k-1)n + r, \;\; i,p,k,r = 1, \dots, n,$$

i.e.

$$N_i^r(x,y) = \Gamma_p{}^r{}_i(x)\overset{1}{v}{}^p, \;\; N_i^{n+r}(x,y) = \Gamma_p{}^r{}_i(x)\overset{2}{v}{}^p, \dots .$$

This $H_{\overset{n}{\oplus}\tau_M}$ is homogeneous, for the connection coefficients defined by (11) are so. We form equivalence classes in the fibres of $\overset{n}{\oplus}\tau_M$. Let $a\colon(\overset{1}{e}, \dots, \overset{n}{e}) \mapsto (\overset{1}{v}, \dots, \overset{n}{v}) = y_0$ be a linear transformation which takes an orthonormal base $\overset{1}{e}, \dots, \overset{n}{e}$ of $\mathbb{R}^n$ endowed with a euclidean metric into the conjugate axis $\overset{1}{v}, \dots, \overset{n}{v}$ of an ellipsoid. Let $f \in O^+(n,\mathbb{R})$ be an orientation preserving rotation of $\mathbb{R}^n$. We consider

$$a_f = a \circ f \circ a^{-1}\colon \overset{n}{\oplus}T_xM \to \overset{n}{\oplus}T_xM$$

and

$$\mathcal{A}_f = \{a_f \mid f \in O^+(n,\mathbb{R})\}$$

and we define the equivalence class of y_0 as $\mathcal{A}_f y_0 = Y$. Then the $(\overset{1}{\tilde{v}}, \dots, \overset{n}{\tilde{v}}) = y \in Y$ form all conjugate-axis systems of the ellipsoid determined by the conjugate axis $(\overset{1}{v}, \dots, \overset{n}{v})$.

We consider the set $\{Y\}$ whose elements represent ellipsoids. The set $\{Y\}$ can be given a (natural) manifold structure (each Y can be identified with an ellipsoid and this with the coefficients g_{ik} of its equation which correspond to a point of $\mathbb{R}^{n^2}$). Thus $\{Y\}$ becomes a manifold $\mathcal{Y}$, and we consider the fiber bundle $Z^n\tau_M = (Z^nTM, \pi, \mathcal{Y}, M)$.

Acccording to (11) $H_{\overset{n}{\oplus}\tau_M}$ acts in case of parallel translation on the components $\overset{\alpha}{v}$ of a $y = (\overset{1}{v}, \dots, \overset{n}{v})$ just as Γ. But Γ takes an ellipsoid by parallel translation into an ellipsoid again, and takes every conjugate axis system of the first ellipsoid into a conjugate axis system of the image ellipsoid. This means that $H_{\overset{n}{\oplus}\tau_M}$ preserves equivalence classes. Hence it induces a connection $H_{Z^n\tau_M}$ in $Z^n\tau_M$.

If $H_{Z^n\tau_M}$ is integrable for one $Y(x_0)$ at least, then the parallel translated Y of $Y(x_0)$ by $H_{Z^n\tau_M}$ are independent of the route γ and depend on the point x alone. To such a Y corresponds in $\overset{n}{\oplus}\tau_M$ an equivalence class which is represented by an ellipsoid, and then the parallel translated of such an ellipsoid by $H_{\overset{n}{\oplus}\tau_M}$ corresponding to $H_{Z^n\tau_M}$ depend also on the point x alone. But this means that the coefficients $g_{ik}(x)$ of these ellipsoids are covariant constant and yield a metrization of Γ.

Thus we have obtained

Theorem 2. *A torsion free linear connection is metrizable iff the above determined $H_{Z^n\tau_M}$ is integrable for a $Y(x_0)$.*

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