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ON KY FAN'S INEQUALITY

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Abstract: The well known Ky Fan inequality

$$(*)\qquad \prod_{i=1}^{n}(x_i/(1-x_i))^{1/n}\leq\sum_{i=1}^{n}x_i/\sum_{i=1}^{n}(1-x_i)$$

is valid for all real numbers $x_i\in(0,1/2]$, $i=1,\ldots,n$. In this paper we present a new proof, a sharpening and several related inequalities of $(*)$.

1. Introduction

The following celebrated inequality, which is due to Ky Fan, is a remarkable counterpart of the famous arithmetic mean-geometric mean inequality.

Ky Fan's Theorem. *If A_n and G_n (resp. A'_n and G'_n) designate the weighted arithmetic and geometric means of the real numbers $x_1,\ldots,x_n$ (resp. $1-x_1,\ldots,1-x_n$) with $x_i\in(0,1/2]$, $i=1,\ldots,n$, i.e.*

$$A_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \quad and \quad G_n = \prod_{i=1}^{n} x_i^{p_i/P_n}$$

$$\left(resp. \; A'_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i(1-x_i) \quad and \quad G'_n = \prod_{i=1}^{n} (1-x_i)^{p_i/P_n} \right)$$

with $P_n = \sum_{i=1}^{n} p_i$ *and positive weights* $p_1 \dots, p_n$, *then*

$$G_n/G'_n \le A_n/A'_n \tag{1.1}$$

with equality holding if and only if $x_1 = \ldots = x_n$.

Inequality (1.1) (with $p_1 = \ldots = p_n = 1$) had been published for the first time in 1961 in the well-known book "Inequalities" by E. F. Beckenbach and R. Bellman [4, p. 5]. Since then Fan's result has received a lot of attention and many interesting new proofs, extensions, sharpenings and variants have been discovered; see [5] and the references therein. The aim of this paper is to present some new results concerning Fan's inequality. In Section 2 we provide a new proof; a sharpening of (1.1) (with $p_1 = \ldots = p_n = 1$) is given in Section 3, and certain results related to Fan's inequality will be presented in the final section.

In what follows we maintain the notations introduced in this section. Further we denote by a_n and g_n (resp. a'_n and g'_n) the unweighted arithmetic and geometric means of $x_1, \ldots, x_n$(resp. $1-x_1, \ldots, 1-x_n$) with $x_i \in (0, 1/2]$, $i = 1, \ldots, n$.

2. A new proof

In [7] one can find what G. H. Hardy, J. E. Littlewood and G. Pólya call "the most familiar of all proofs" [7, p. 19] of the arithmetic mean-geometric mean inequality, which "is due (so far as we have been able to trace it) to Maclaurin" [7, p. 19]. We use this technique (which is explained in detail in [7, §2.6]) in order to provide a new proof of Fan's Theorem.

If $x_1 = \ldots = x_n$, then we obtain obviously

$$G_n/G'_n = A_n/A'_n .$$

Next we assume that the numbers $x_i \in (0, 1/2]$, $i = 1, \ldots, n$, are not all equal. Without loss of generality we may suppose $x_1 \neq x_2$. We denote by f the function

$$f : [0,1] \to \mathbb{R}\,,$$
$$f(p) = \left(\frac{x_1}{1-x_1}\right)^p \left(\frac{x_2}{1-x_2}\right)^{1-p} \frac{p(1-x_1)+(1-p)(1-x_2)}{px_1+(1-p)x_2}\,.$$

A simple calculation yields that $\log(f)$ is strictly convex and since $f(0) = f(1) = 1$ we conclude $f(p) < 1$ for $p \in (0,1)$, and if we set $p = p_1/(p_1+p_2)$, then the last inequality is equivalent to

$$\text{(2.1)} \quad \left(\frac{x_1}{1-x_1}\right)^{p_1} \left(\frac{x_2}{1-x_2}\right)^{p_2} < \left[\frac{p_1x_1+p_2x_2}{p_1(1-x_1)+p_2(1-x_2)}\right]^{p_1+p_2}.$$

Next we define

$$g(x_1,\ldots,x_n) = \prod_{i=1}^{n} (x_i/(1-x_i))^{p_i}\,,$$

then we obtain from (2.1) for all $x_i \in (0,1/2]$, $i = 1,\ldots,n$:

$$\text{(2.2)} \quad g(x_1,\ldots,x_n) < g\left(\frac{p_1x_1+p_2x_2}{p_1+p_2}, \frac{p_1x_1+p_2x_2}{p_1+p_2}, x_3,\ldots,x_n\right).$$

Let us denote the maximum of the set

$$\left\{g(y_1,\ldots,y_n) | 0 \le y_i \le 1/2\,,\ i = 1,\ldots,n\,,\ \sum_{i=1}^{n} p_i y_i = \sum_{i=1}^{n} p_i x_i\right\}$$

by $g(y_1^*,\ldots,y_n^*)$. Since $y_i^* \neq 0$, $i = 1,\ldots,n$, we conclude from (2.2) $y_1^* = y_2^* = \ldots = y_n^*$, and hence we have

$$y_1^* = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\,.$$

Finally we get

$$\prod_{i=1}^{n} (x_i/(1-x_i))^{p_i} < \prod_{i=1}^{n} (y_i^*/(1-y_i^*))^{p_i} = (A_n/A_n')^{P_n}\,,$$

which we had to show. $\Diamond$

3. A sharpening of Fan's inequality

In this section we present a refinement of inequality (1.1) for the special case $p_1 = \ldots = p_n = 1$.

Theorem 1. *If* $x_i \in (0,1/2]$, $i = 1,\ldots,n$ $(n \ge 2)$, *then*

$$(3.1) \qquad g_n/g_n' \le \prod_{i=1}^{n}\left(\sum_{\substack{j=1\\j\neq i}}^{n} x_j \Big/ \sum_{\substack{j=1\\j\neq i}}^{n}(1-x_j)\right)^{1/n} \le a_n/a_n' .$$

If $n = 2$, then the left-hand side of (3.1) is an identity. Otherwise, equality holds if and only if $x_1 = \ldots = x_n$.

Proof. For $n \ge 3$ we conclude from Fan's inequality:

$$\prod_{i=1}^{n}\frac{x_i}{1-x_i} = \prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(\frac{x_j}{1-x_j}\right)^{1/(n-1)} \le \prod_{i=1}^{n}\left(\sum_{\substack{j=1\\j\neq i}}^{n} x_j \Big/ \sum_{\substack{j=1\\j\neq i}}^{n}(1-x_j)\right),$$

where the sign of equality holds if and only if we have for every $i \in \in \{1,\ldots,n\}$:

$$x_1 = x_2 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_n ,$$

i.e. $x_1 = \ldots = x_n$. Let $n \ge 2$; we set

$$s_i = \frac{1}{n-1}\sum_{\substack{j=1\\j\neq i}}^{n} x_j , \quad i = 1,\ldots,n;$$

then $s_i \in (0, 1/2]$, $i = 1,\ldots,n$, and we obtain from Fan's inequality:

$$\prod_{i=1}^{n}\left(\sum_{\substack{j=1\\j\neq i}}^{n} x_j \Big/ \sum_{\substack{j=1\\j\neq i}}^{n}(1-x_j)\right)^{1/n} = \prod_{i=1}^{n}\left(\frac{s_i}{1-s_i}\right)^{1/n} \le$$

$$\le \sum_{i=1}^{n} s_i \Big/ \sum_{i=1}^{n}(1-s_i) = \sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n}(1-x_i),$$

where equality is valid if and only if $s_1 = \ldots = s_n$ i.e. $x_1 = \ldots = x_n$. ◊

We have tried unsuccessfully to extend (3.1) to weighted mean values in order to provide a sharpening of inequality (1.1). We note that a similar (but different) result of (3.1) for weighted means is given in [10].

4. Related inequalities

In [2] the following converse of inequality (1.1) (with $p_1 = \ldots = = p_n = 1$) has been proved: If $x_i \in (0,1)$, $i = 1,\ldots,n$, *are real numbers, then*

$$\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} (1-x_i) \le \prod_{i=1}^{n} (x_i/(1-x_i))^{x_i/\sum_{i=1}^{n} x_j} \tag{4.1}$$

with equality holding if and only if $x_1 = \ldots = x_n$. Here we give a generalization and a counterpart of inequality (4.1).

Theorem 2. *Let* $x_i \in (0,1)$, $i = 1, \ldots, n$, *be real numbers. If* $p_i \ge 0$, $i = 1, \ldots, n$, *are real numbers which are not all equal to* 0, *then we have for* $\lambda > 0$ *and* $\nu > 0$:

$$\begin{aligned} &\prod_{i=1}^{n} \left(x_i^{\lambda}/(1-x_i)^{\nu}\right)^{p_i(1-x_i)/\sum_{j=1}^{n} p_j(1-x_j)} \le \\ &\le (A_n)^{\lambda}/(A'_n)^{\nu} \le \prod_{i=1}^{n} \left(x_i^{\lambda}/(1-x_i)^{\nu}\right)^{p_i x_i/\sum_{j=1}^{n} p_j x_j} . \end{aligned} \tag{4.2}$$

For positive p_i, $i = 1, \ldots, n$, *the equalities are valid if and only if* $x_1 = \ldots = x_n$.

Proof. We define for $x \in (0,1)$:

$$f(x) = \log(x^{\lambda}/(1-x)^{\nu})^{x} \quad \text{and} \quad g(x) = \log(x^{\lambda}/(1-x)^{\nu})^{1-x} .$$

Because of

$$f''(x) = \frac{\lambda}{x} + \frac{\nu}{1-x} + \frac{\nu}{(1-x)^2} > 0 \quad \text{and} \quad g''(x) = -\frac{\lambda}{x} - \frac{\nu}{1-x} - \frac{\lambda}{x^2} < 0$$

we conclude that f and g are respectively strictly convex and strictly concave, and double-inequality (4.2) is an immediate consequence of Jensen's inequality for f and g. ◊

Remark. Setting $\lambda = \nu = p_1 = \ldots = p_n = 1$ in the second inequality of (4.2) we obtain (4.1).

The following inequality, due to P. Henrici [8], had been published in 1956: *If* $y_i \ge 1$, $i = 1, \ldots, n$, *are real numbers, then*

$$\frac{n}{1 + \prod_{i=1}^{n} y_i^{1/n}} \le \sum_{i=1}^{n} \frac{1}{1+y_i} . \tag{4.3}$$

If $0 < y_i \le 1$, $i = 1, \ldots, n$, *then inequality* (4.3) *is reversed. Equality holds if and only if* $y_1 = \ldots = y_n$.

Concerning extensions of Henrici's theorem we refer to [5], [9] and the references therein. Recently, J. Sándor [11] made the interesting discovery that inequality (4.3) and Fan's inequality (1.1) (with $p_1 = \ldots = p_n = 1$) are equivalent. Our next proposition presents a slight extension of Sándor's result to weighted means.

Theorem 3. *The two inequalities*

$$G_n/G'_n \le A_n/A'_n \quad (0 < x_i \le 1/2\,,\ i = 1, \dots, n) \tag{4.4}$$

and

$$\frac{P_n}{1 + \prod_{i=1}^{n} y_i^{p_i/P_n}} \le \sum_{i=1}^{n} \frac{p_i}{1 + y_i} \quad (y_i \ge 1\,,\ i = 1, \dots, n) \tag{4.5}$$

are equivalent.

Proof. If we set $x_i = 1/(1 + y_i)$, $i = 1, \dots, n$, in (4.4), then we obtain (4.5); and if we put $y_i = (1 - x_i)/x_i$, $i = 1, \dots, n$, in (4.5), then we get (4.4). $\Diamond$

Because of the equivalence of (4.4) and (4.5) and since Henrici's inequality was published five years before Fan's inequality, Sándor proposes to call (1.1) the inequality of Henrici-Fan. We note that already in 1943 P. Kesava Menon discovered the following generalization of Henrici's theorem for the special case $p_1 = \ldots = p_n = 1$: *If a and $y_1, \dots, y_n$ are positive real numbers and if $t \ne 0$ is a real number such that $y_i \ge a/t$, $i = 1, \dots, n$, then*

$$\frac{P_n}{\left(a + \prod_{i=1}^{n} y_i^{p_i/P_n}\right)^t} \le \sum_{i=1}^{n} \frac{p_i}{(a + y_i)^t}\,. \tag{4.6}$$

If $y_i \le a/t$, $i = 1, \dots, n$, then the inequality is reversed. (We remark that the inequality given in [9, p. 284] is stated incorrectly.) This proposition follows from applying Jensen's inequality to the function $x \mapsto (a + e^x)^{-t}$ and then replacing e^{x_i} by y_i, $i = 1, \dots, n$.

An application of inequality (4.6) leads to a new extension of (1.1).

Theorem 4. *Let t be a real number with $0 < t \le 1$. Then we have for $x_i \in (0, t/(t+1)]$, $i = 1, \dots, n$:*

$$G_n/G'_n \le t \sum_{i=1}^{n} p_i x_i^t \Big/ \sum_{i=1}^{n} p_i (1 - x_i^t) \tag{4.7}$$

with equality if and only if $t = 1$ and $x_1 = \ldots = x_n$.

Proof. Setting $a = 1$ and $y_i = (1 - x_i)/x_i$, $i = 1, \dots, n$, we obtain $y_i \ge 1/t$, $i = 1, \dots, n$, and from (4.6) we conclude

$$P_n(1+G_n'/G_n)^{-t} \le \sum_{i=1}^n p_i x_i^t$$

which is equivalent to

$$\left(1+\sum_{i=1}^n p_i(1-x_i^t)/\sum_{i=1}^n p_i x_i^t\right)^{1/t} \le 1+G_n'/G_n\,. \tag{4.8}$$

Using the generalized Bernoulli inequality [9, p. 34] we get a lower bound for the left-hand side of (4.8):

$$1+\sum_{i=1}^n p_i(1-x_i^t)/t\sum_{i=1}^n p_i x_i^t \le \left(1+\sum_{i=1}^n p_i(1-x_i^t)/\sum_{i=1}^n p_i x_i^t\right)^{1/t}. \tag{4.9}$$

Equality holds in (4.9) if and only if $t=1$. From (4.8) and (4.9) we obtain immediately inequality (4.7). If equality is valid in (4.7), then we conclude that $t=1$ and from Fan's theorem we get $x_1=\ldots=x_n$. ◊

In 1984 W.-L. Wang and P.-F. Wang [12] proved a noteworthy counterpart of Fan's inequality: *If H_n (resp. H_n') denotes the weighted harmonic mean of $x_1,\ldots,x_n$ (resp. $1-x_1,\ldots,1-x_n$) with $x_i\in(0,1/2]$, $i=1,\ldots,n$, i.e.*

$$H_n=P_n/\sum_{i=1}^n p_i/x_i \quad (\textit{resp. } H_n'=P_n/\sum_{i=1}^n p_i/(1-x_i))\,,$$

then

$$H_n/H_n' \le G_n/G_n' \tag{4.10}$$

with equality holding if and only if $x_1=\ldots=x_n$. We note that Wang/Wang proved inequality (4.10) for the special case $p_1=\ldots=$ $=p_n=1$. A proof for (4.10) can be found in [3].

In what follows we present an extension of the double-inequality

$$H_n/H_n' \le G_n/G_n' \le A_n/A_n'\,. \tag{4.11}$$

We define for real values α and $x_i\in(0,1/2]$, $i=1,\ldots,n$:

$$A_{n,\alpha}=\frac{1}{P_n}\sum_{i=1}^n p_i x_i^\alpha/[x_i^\alpha+(1-x_i)^\alpha]$$

and

$$H_{n,\alpha} = P_n / \sum_{i=1}^{n} p_i [x_i^\alpha + (1-x_i)^\alpha]/x_i^\alpha .$$

$A'_{n,\alpha}$ and $H'_{n,\alpha}$ will be obtained from $A_{n,\alpha}$ and $H_{n,\alpha}$ by replacing x_i by $1-x_i$, $i=1,\dots,n$. Then we have

Theorem 5. *If $\alpha > 0$ and $x_i \in (0,1/2]$, $i=1,\dots,n$, then*

$$H_{n,\alpha}/H'_{n,\alpha} \le (G_n/G'_n)^\alpha \le A_{n,\alpha}/A'_{n,\alpha} \tag{4.12}$$

with equality holding if and only if $x_1 = \ldots = x_n$.

Proof. If we replace in (4.11) the values x_i by $x_i^\alpha/[x_i^\alpha + (1-x_i)^\alpha]$, $i=1,\dots,n$, then we get (4.12). ◊

Remark. Setting $\alpha = 1$ in (4.12) we obtain (4.11) (see [11]).

In [1] a proof for the inequality

$$g_n - g'_n \le a_n - a'_n \tag{4.13}$$

is given. Inequality (4.13) is an additive counterpart of $g_n/g'_n \le a_n/a'_n$.

In the last section of this paper we want to give an additive counterpart of $(g_n/g'_n)^n \le (a_n/a'_n)^n$. The following surprising proposition holds:

Theorem 6. *If $x_i \in (0,1/2]$, $i=1,\dots,n$, then*

$$(a_n)^n - (a'_n)^n \le (g_n)^n - (g'_n)^n . \tag{4.14}$$

If $n=1,2$ or if $x_1 = \ldots = x_n$, then equality holds in (4.14). *If $n \ge 3$, then equality is valid if and only if $x_1 = \ldots = x_n$.*

Proof. The main tool of our proof is the following intriguing identity due to A. Dinghas [6]:

$$(a_n)^n - (g_n)^n = \sum_{k=2}^{n} \frac{1}{k^2}(x_k - a_{k-1})^2 Q_{k-2}(a_k, a_{k-1}) \prod_{i=k+1}^{n} x_i$$

where $Q_0(x,y) = 1$ and

$$Q_{k-2}(x,y) = x^{k-2} + 2x^{k-3}y + \ldots + (k-1)y^{k-2}, \ k \ge 3 .$$

This leads to

$$[(a_n)^n - (g_n)^n] - [(a'_n)^n - (g'_n)^n] = \sum_{k=2}^{n} \frac{1}{k^2}(x_k - a_{k-1})^2 \cdot \tag{4.15}$$

$$\cdot \left[Q_{k-2}(a_k, a_{k-1}) \prod_{i=k+1}^{n} x_i - Q_{k-2}(1-a_k, 1-a_{k-1}) \prod_{i=k+1}^{n} (1-x_i) \right] .$$

Since $Q_{k-2}(x,y)$ ($k \ge 3$, $x \ge 0$, $y \ge 0$) is strictly increasing in x and y we conclude that the term in square brackets is non-positive which proves (4.14). If $n = 1, 2$ or if $x_1 = \ldots = x_n$, then equality holds in (4.14) obviously. We assume

$$0 < x_1 < x_2 \leq x_3 \leq \ldots \leq x_n \leq 1/2 \quad \text{and} \quad n \geq 3 .$$

If equality holds in (4.14), then each term in the sum of (4.15) must be equal to 0. For $k = 2$ we get:

$$\frac{1}{4}(x_2 - x_1)^2 \left[\prod_{i=3}^{n} x_i - \prod_{i=3}^{n} (1 - x_i) \right] = 0$$

which implies $x_3 = \ldots = x_n = 1/2$. For $k = 3$ we obtain

$$\frac{1}{9}(x_3 - a_2)^2 [Q_1(a_3, a_2) - Q_1(1 - a_3, 1 - a_2)](1/2)^{n-3} = 0 .$$

Since $a_2 < x_2 \leq x_3$ we have

$$Q_1(a_3, a_2) = Q_1(1 - a_3, 1 - a_2)$$

and hence $a_2 = 1 - a_2$, i.e. $x_1 = x_2 = 1/2$. Thus, if $n \geq 3$, then equality holds in (4.14) if and only if $x_1 = \ldots = x_n$. This completes the proof of Th. 6. $\Diamond$

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