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ON THE AREA SUM OF A CONVEX POLYGON AND ITS POLAR RECIPROCAL

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To o. Univ.-Prof. Dr. H. Vogler on his 60th birthday

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Abstract: Let P be a plane convex polygon contained in the unit circle K, and let $P^\star$ be the polar reciprocal of P with respect to K. In this paper it is proved that the area sum of P and $P^\star$ is greater than or equal to 6 with equality if and only if P is a square inscribed in K.

1. Introduction

Let K be the unit circle centred at the origin O, and let P be a convex polygon inscribed in K and containing O in its interior. We denote by $P^\star$ the circumscribed polygon whose points of contact with K are the vertices of P. J. Aczél and L. Fuchs [1] proved that

$$a(P) + a(P^\star) \geq 6, \tag{1}$$

where $a(X)$ denotes the area of the set X. Equality holds if and only if P is a square. An alternative proof was given by E. Trost [5]. Complementary remarks to (1) were made by J. Rätz [4]. More genereally, L. Kuipers and B. Meulenbeld [3] found the infimum of the weighted area sum $wa(P) + (1-w)a(P^\star)$ for any weight w between 0 and 1, the infimum depending on w. They also obtained a similar result for the weighted perimeter sum of P and $P^\star$.

In the present paper we shall extend inequality (1) to more general domains.

Theorem. *Let P be a convex polygon contained in the unit circle K. If $P^\star$ is the polar reciprocal domain of P with respect to K, then inequality* (1) *holds and equality occurs only if P is a square inscribed in K.*

2. Proof of the Theorem

We begin with a further proof of the theorem by Aczél and Fuchs. Let P be a convex polygon inscribed in K, and let $P^\star$ be the polar reciprocal domain of P. We may assume that P contains the centre O of K in its interior, since otherwise $a(P^\star) = \infty$. Let us denote the central angles spanned by the sides of P by $2x_1, \ldots, 2x_n$, where

$$(2) \qquad \begin{gathered} 0 < x_1 \le x_2 \le \ldots \le x_n < \pi/2, \\ x_1 + \ldots + x_n = \pi. \end{gathered}$$

If the function f is defined by

$$f(x) = \sin x \cos x + \tan x,$$

we have to show that

$$(3) \qquad S \equiv \sum_{i=1}^{n} f(x_i) \ge 6$$

with equality only for $n = 4$ and $x_1 = x_2 = x_3 = x_4 = \pi/4$.

From

$$f'(x) = 2\cos^2 x - 1 + \frac{1}{\cos^2 x}$$

and

$$f''(x) = 2\frac{\sin x}{\cos^3 x}\left(1 - 2\cos^4 x\right)$$

we see that (i) f is strictly increasing in $0 \le x < \pi/2$; (ii) strictly concave in $0 \le x \le x_0$, and convex in $x_0 \le x < \pi/2$, where

$$x_0 = \arccos 1/\sqrt{[4]2} = 32.765\ldots^\circ .$$

In the proof of (3) we may assume that

$$(4) \qquad x_0 \le x_2.$$

If, on the contrary, $0 < x_1 \le x_2 < x_0$, we can replace x_1 and x_2 by x_1' and x_2' such that

$$0 \le x_1' < x_1 \le x_2 < x_2' \le x_0,$$

$$x_1' + x_2' = x_1 + x_2$$

and $x_1' = 0$ or $x_2' = x_0$ or both. Since f is strictly concave in $[0, x_0]$, this process reduces the sum S. Moreover, the number of the x_i's contained in $(0, x_0)$ would decrease. After a finite number of steps we obtain a finite set of points, again denoted by $\{x_1, \ldots, x_n\}$, which satisfies (2) and (4) and yields a smaller S.

We now show that S can be diminished by displacing x_1 if

$$0 < x_1 < x_0 \leq x_2 \leq \ldots \leq x_n < \pi/2. \tag{5}$$

Since f is strictly convex in $[x_0, \pi/2)$, we have

$$S \geq f(x_1) + (n-1) f\Big(\frac{\pi - x_1}{n-1}\Big) \equiv S(x_1) \tag{6}$$

with equality only if $x_2 = \ldots = x_n = (\pi - x_1)/(n-1)$. By (5), we note that $(n-1)x_0 < \pi$, whence

$$n \leq 6.$$

From (6) it follows that

$$S'(x_1) = \Big(\cos^2 x_1 - \cos^2 \frac{\pi - x_1}{n-1}\Big)\Big(2 - \cos^{-2} x_1 \cos^{-2} \frac{\pi - x_1}{n-1}\Big). \tag{7}$$

We now distinguish the following cases:

$n = 3$ or 4. For $0 < x_1 < x_0$ we have

$$\frac{\pi}{2} > \frac{\pi - x_1}{n-1} > \frac{\pi - x_0}{3} > x_0\,,$$

which shows that

$$\cos^2 x_1 - \cos^2 \frac{\pi - x_1}{n-1} > 0,$$

and

$$\cos \frac{\pi - x_1}{n-1} < \cos \frac{\pi - x_0}{3} = 0.655\ldots < \frac{1}{\sqrt{2}},$$

whence

$$2 - \cos^{-2} x_1 \cos^{-2} \frac{\pi - x_1}{n-1} < 0.$$

Thus

$$S'(x_1) < 0$$

and

$$S(x_1) > S(x_0) \tag{8}$$

if $x_1 < x_0$.

$n = 5$ or 6. The function g defined by

$$g(x_1) \equiv 2\cos x_1 \cos\frac{\pi - x_1}{n-1} = \cos\left(\frac{\pi - x_1}{n-1} + x_1\right) + \cos\left(\frac{\pi - x_1}{n-1} - x_1\right)$$

has the derivatives

$$g'(x_1) = -\left(1 - \frac{1}{n-1}\right)\sin\left(\frac{\pi - x_1}{n-1} + x_1\right) + \\ + \left(1 + \frac{1}{n-1}\right)\sin\left(\frac{\pi - x_1}{n-1} - x_1\right),$$

$$g''(x_1) = -\left(1 - \frac{1}{n-1}\right)^2\cos\left(\frac{\pi - x_1}{n-1} + x_1\right) - \\ - \left(1 + \frac{1}{n-1}\right)^2\cos\left(\frac{\pi - x_1}{n-1} - x_1\right).$$

In view of $\frac{\pi - x_1}{n-1} < \frac{\pi}{4}$ and $x_1 < x_0 < \frac{\pi}{4}$ we have $g''(x_1) < 0$ so that g is positive and strictly concave on $[0, x_0]$. This implies that $\cos^{-2} x_1 \cos^{-2}\frac{\pi - x_1}{n-1}$ is strictly convex and

$$h(x_1) = 2 - \cos^{-2} x_1 \cos^{-2}\frac{\pi - x_1}{n-1}$$

is strictly concave in $[0, x_0]$.

$n = 5$. Since $h(x_1) > 0$ for x_1 close to 0, and $\cos^2\frac{\pi - x_0}{4} < < \cos^2 x_0 = \frac{1}{\sqrt{2}}$, the function h passes from positive to negative values on $(0, x_0]$. By (7), S' and h have the same sign, since $x_1 < \frac{\pi - x_1}{4}$ on $[0, x_0]$. Hence S attains its minimum only at one of the end points of the interval $[0, x_0]$. The fact that $S(0) = 4f(\frac{\pi}{4}) = 6$ and $S(x_0) = 6.010\ldots$ shows that

$$S(x_1) > S(0) \tag{9}$$

for $x_1 > 0$.

$n = 6$. The supposition (5) restricts the variable x_1 to

$$0 < x_1 \leq \pi - 5x_0,$$

where $\pi - 5x_0 < x_0$. Since h is strictly concave on $[0, \pi - 5x_0]$, $h(0) = = 1 - \tan^2(\pi/5) > 0$ and $h(\pi - 5x_0) = 2 - \cos^{-2} x_0 \cos^{-2}(\pi - 5x_0) > > 2 - \cos^{-4} x_0 = 0$ we conclude that $h(x_1) > 0$ for $x_1 > 0$. Because $(\pi - x_1)/5 \geq x_0 > x_1$, we have

$$\cos^2 x_1 - \cos^2\frac{\pi - x_1}{5} > 0.$$

By (7), this shows that $S'(x_1) > 0$ and (9) is satisfied once more.

In conclusion, we state that

$$S \geq \inf \, mf\Big(\frac{\pi}{m}\Big) \tag{10}$$

for $m = 3, 4, \dots$, where $\pi/m \geq x_0$. But $m \leq \pi/x_0$ implies that $m = 3, 4$ or 5. The required inequality (3) follows from $3f(\pi/3) = 15\sqrt{3}/4 = 6.495\dots$, $4f(\pi/4) = 6$ and $5f(\pi/5) = 6.010\dots$.

Let P be a convex polygon contained in the unit circle K with centre O. To prove inequality (1) we may assume that O is an interior point of P, since otherwise $a(P^\star) = \infty$. Let $n \geq 3$ be given. By a convex n-gon we mean a convex polygon with at most n sides. There exists a convex n-gon P contained in K and containing O in its interior and having the property that $a(P) + a(P^\star)$ attains its minimum. The proof of our theorem is completed by the following lemma.

Lemma. *All the vertices of P are on the boundary of K.*

Proof. Let $P = A_1A_2 \dots A_n$ and $P^\star = B_1B_2 \dots B_n$ be such that [4] $B_i \vee B_{i+1}$ is the polar of A_i, for $i = 1, \dots, n$. Suppose that A_1 is an inner point of K. Then $B_1 \vee B_2$ does not intersect K. We denote the interior angles of $P^\star$ at B_1 and B_2 by β_1 and β_2 respectively and distinguish the following two cases.

$\beta_1 + \beta_2 > \pi$. The lines $B_n \vee B_1$ and $B_3 \vee B_2$ intersect outside $P^\star$ at a point U which is the pole of $A_2 \vee A_n$. The segment joining O and U intersects B_1B_2 at an inner point T. The polar t of T is parallel to $A_2 \vee A_n$ and contains the vertex A_1. Since $\overline{OT} < \overline{OU}$, the line $A_2 \vee A_n$ separates O and A_1. Without loss of generality, we may assume that $\overline{B_1T} \leq \overline{TB_2}$. We displace A_1 on t through a small distance and obtain a new convex n-gon $P' = A_1'A_2 \dots A_n$ contained in K. The polar n-gon $P'^\star = B_1'B_2'B_3 \dots B_n$ arises from $P^\star$ by rotating $B_1 \vee B_2$ about T. We choose the direction of the displacement of A_1 so that B_2' lies on B_2B_3 and B_1' on the elongated segment B_nB_1. Let p be the ray radiating from B_2, parallel to $B_n \vee B_1$ and intersecting the interior of $P^\star$ (this is possible because $\beta_1 + \beta_2 > \pi$). The segment $B_1'B_2'$ intersects p at a point B_2''. Then

$$\overline{B_1'T} \leq \overline{TB_2''} < \overline{TB_2'},$$

whence

$$a(TB_1B_1') < a(TB_2B_2')$$

and

$$a(P'^\star) < a(P^\star).$$

Since $a(P') = a(P)$, we have a contradiction to the assumption that $a(P) + a(P^\star)$ is minimal.

$\beta_1 + \beta_2 \le \pi$. By displacing A_1 on the ray OA_1 towards the boundary of K through a small distance x we obtain a new convex n-gon $P' = A_1' A_2 \dots A_n$. Let b be length of the orthogonal projection of $A_2 A_n$ onto the perpendicular to $O \vee A_1$. Then

$$a(P') - a(P) = a(A_1 A_1' A_n) + a(A_1 A_1' A_2),$$

whence

$$\frac{1}{x}(a(P') - a(P)) = \frac{1}{2} b.$$

In view of $b \le \overline{A_2 A_n} \le 2$ this implies

$$\frac{1}{x}(a(P') - a(P)) \le 1. \tag{11}$$

If $\overline{OA_1} = d$, the polar of A_1' has the distance $1/(d+x)$ from O. Thus the polar n-gon of P', $P'^\star = B_1' B_2' B_3 \dots B_n$, arises from $P^\star$ by displacing the side $B_1 B_2$ parallel to itself towards O through the distance

$$\frac{1}{d} - \frac{1}{d+x} = \frac{x}{d(d+x)}.$$

Hence

$$\begin{aligned} a(P^\star) - a(P'^\star) &= a(B_1 B_2 B_2' B_1') \\ &= (\overline{B_1 B_2} + \overline{B_1' B_2'}) \cdot x/2d(d+x). \end{aligned}$$

But clearly

$$\overline{B_1 B_2} \ge \cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} \ge \cot \frac{\beta_1}{2} + \tan \frac{\beta_1}{2} \ge 2$$

and also $\overline{B_1' B_2'} \ge 2$. Since $d < 1$ and $d + x \le 1$, we finally have

$$\frac{1}{x}(a(P^\star) - a(P'^\star)) > 2.$$

The combination with (11) yields

$$a(P') + a(P'^\star) < a(P) + a(P^\star)$$

which is impossible. Thus the lemma and the theorem are proved. $\Diamond$

Corollary. *Let C be a closed convex set contained in the unit circle K and let $C^\star$ be its polar reciprocal. Then*

$$a(C) + a(C^\star) \ge 6. \tag{12}$$

If C is contained in the interior of K, then strict inequality holds.

Proof. It suffices to consider a closed convex subset C of K having the centre O of K as an inner point. The sets C and $C^\star$ can be approximated by pairs of polar reciprocal convex polygons. Therefore, (12) is a consequence of the theorem. For any $r \in (0,1)$, the set rC is in the interior of K, and $(rC)^\star = \frac{1}{r}C^\star$. The function

$$f(r) = a(rC) + a(\frac{1}{r}C^\star) = r^2 a(C) + \frac{1}{r^2} a(C^\star)$$

has a negative derivative

$$f'(r) = \frac{2}{r^3}(r^4 a(C) - a(C^\star)) < 0.$$

Hence

$$f(r) > f(1) = a(C) + a(C^\star) \geq 6,$$

as required. $\Diamond$

3. Remarks

(i) It may be that in (12) equality holds only if C is a square inscribed in K.

(ii) In the corollary, the assumption of convexity of C is essential. If C is the boundary of K, then $a(C) + a(C^\star) = \pi$.

(iii) In Euclidean 3-space let K be a solid unit sphere, P a convex polyhedron inscribed in K and $P^\star$ the polar reciprocal of P with respect to K. In the following list the values of $V(P) + V(P^\star)$ are collected, where P is a regular polyhedron (characterized by its number n of vertices), and V the volume

n	$V(P) + V(P^\star)$
4	14.36960...
6	9.33333...
8	8.46780...
12	8.08644...
20	7.83921...

and $V(K) + V(K^\star) = 8.37758\ldots$ The infimum of $V(P) + V(P^\star)$, extended over all convex polyhedra P inscribed in K, remains unknown and is not attained by the cube or the regular octahedron. In place of the volume, various other functionals may be considered. A simple example is given by the mean width $M(C)$ of a convex body C in

$E^d (d \geq 2)$, i.e. the mean value of the widths of C, taken over all possible directions in E^d. Let the origin O be an interior point of a body C which need not necessarily be a subset of K. W. Firey observed that $\frac{(C+C^\star)}{2} \supset K$ (formula (1) in [2]). This implies that

$$M(C) + M(C^\star) \geq 4$$

with equality only if $C = K$. However, if O is not an interior point of C, than $C^\star$ is unbounded.

References

[1] ACZÉL, J. and FUCHS, L.: A minimum-problem on areas of inscribed and circumscribed polygons of a circle, *Compositio Math.* **8** (1950), 61–67.

[2] FIREY, WM. J.: The mixed area of a convex body and its polar reciprocal, *Israel J. Math.* **1** (1963), 201–202.

[3] KUIPERS, L. and MEULENBELD, B.: Two minimum-problems. I, II, III. *Nederl. Akad. Wetensch. Proc. Ser. A* **54** = *Indagationes Math.* **13** (1951), 135–142, 143–151, 237–242.

[4] RÄTZ, J.: On special pairs of polygons with minimal area sum, *Intern. Series Num. Math.* **103** (1992), 455–458.

[5] TROST, E.: Beweis einer Minimaleigenschaft des Quadrates, *Elem. Math.* **6** (1951), 26–28.