

PRODUCTS OF PSEUDORADIAL SPACES

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Abstract: Some recent results concerning the product of pseudoradial spaces are presented in an historical perspective and some new results are proved. It is shown that the product of less than \mathfrak{h} (for $\mathfrak{h} \leq \omega_3$) compact T_2 pseudoradial spaces each of cardinality $< 2^{\omega_2}$ is pseudoradial and that the product of two compact T_2 almost radial spaces is almost radial if one of them is semi-radial.

1. Introduction and basic definitions

We recall here the principal definitions concerning pseudoradial spaces. The class of *pseudoradial* spaces, with the name of *folgenbestimmt Räume* was introduced by H. Herrlich in 1967 (see [15]), taken in consideration on various circumstances with the name of *chain-net* spaces (see [20] and [21]) and considered as an interesting argument of investigation by A.V. Arhangel'skiĭ in 1978 (see [1]). A topological

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space X is said to be *pseudoradial* (or *chain-net* or *folgenbestimmt*) if for any non-closed subset $A \subset X$ there is a point $x \in \overline{A} \setminus A$ and a sequence $(x_\alpha)_{\alpha < \lambda}$ such that $x_\alpha \in A$ and $x_\alpha \rightarrow x$. A space is said to be *radial* (or *Fréchet chain-net* or *stark folgenbestimmt*) if for any $A \subset X$ and any $x \in \overline{A}$ there is a sequence $(x_\alpha)_{\alpha < \lambda}$ converging to x . These classes of spaces are obvious generalizations of the classes of sequential and Fréchet-Urysohn spaces respectively.

By a *sequence* we mean an application from an ordinal number λ to a topological space X . It is easily seen that only regular initial ordinals need to be considered, i.e. regular cardinal numbers. Hereafter the notation of cardinal numbers will be unified with that of the corresponding ordinal, following [19]. So ω_i will be used instead of \aleph_i .

Some subclasses of the class of pseudoradial spaces were introduced successively since they were useful for solving natural problems raised for pseudoradial spaces. First came the class of *almost radial* spaces. We recall that a sequence $(x_\alpha)_{\alpha < \lambda}$ is said to be a *t-sequence* or a *t λ -sequence* if $x_\alpha \rightarrow x$ but $x \notin \{x_\gamma : \gamma < \beta\}$ for any $\beta < \lambda$ (see [10] and [2]). A topological space X is said to be *almost radial* if for any non-closed subset $A \subset X$ there is a *t λ -sequence* with elements in A converging to a point $x \in \overline{A} \setminus A$. The *t* in the name of the sequence came from *target* (Arhangel'skiĭ had introduced the notion of almost radial space through that of *target point* of a set [1]). The *t*-sequences were also called *strict* by Bella and Gerlits [6] and *thin* by Nyikos [22]. They were initially considered in connection with the problem of characterizing sequential spaces in the class of T_1 pseudoradial ones. It was proved [2] that a T_1 space is sequential if and only if it is almost radial and has countable tightness.

Let us now come to products of pseudoradial spaces. At the Prague Topological Symposium in 1986, one of the authors (G.T.) discussed with János Gerlits the problem of the productivity of pseudoradiality. It was clear that, in general, the product of two pseudoradial spaces is not pseudoradial, as already recognized in [15]. However we considered as an interesting subject of investigation to find out sufficient conditions under which pseudoradiality is preserved in a topological product of two or more factors. In the springtime of 1987 Gerlits announced in a letter some very interesting results, which shall be described, together with further developments, in the next section.

2. Principal known results on the product of pseudoradial spaces

The results presented in the private communication of Gerlits and published successively in the Proceedings of the Baku Conference on Topology (1987), were the following [14]

Theorem 2.1. *If X and Y are two compact Hausdorff radial spaces, then their product is pseudoradial.*

Example 2.2. *If $X = [0, 1]$ is the unit interval with the usual Euclidean topology and Y is the one-point Lindelöfization of ω_1 , then X and Y are radial spaces, but their product $X \times Y$ is not pseudoradial.*

This example oriented further work on products of pseudoradial spaces, limiting the search for sufficient conditions to the class of Hausdorff compact pseudoradial spaces. However one result was given in [14] concerning non-compact spaces

Theorem 2.3. *Let X be a pseudoradial Hausdorff countably compact space and Y a sequential space. Then $X \times Y$ is pseudoradial.*

The following result presented in [14] is also interesting

Definition 2.4. A topological space X is said to have *property $\mathcal{P}(\kappa)$* for a regular cardinal number $\kappa \geq \omega$ if for any $A \in [X]^\kappa$ there is a nontrivial sequence in A , converging to some point in X .

Then it was proved

Theorem 2.5. *Let X be a compact Hausdorff space. Then the following are equivalent*

- (a) $X \times Y$ is pseudoradial for every Y pseudoradial;
- (b) X has $\mathcal{P}(\kappa)$ for every regular $\kappa \geq \omega$.

The problems left open by Gerlits and Nagy were: Is it true that the product of two Hausdorff compact pseudoradial spaces is pseudoradial? Is this true if at least one of them is radial?

The first problem is still open (at least in ZFC). The second one was solved positively by Frolík and Tironi [12] who proved the following

Theorem 2.6. *The product of two compact Hausdorff spaces is pseudoradial if one of them is radial.*

If not, by contradiction, there is a couple of compact Hausdorff spaces, say X radial and Y pseudoradial, whose product is not pseudoradial. Then there is some sequentially closed but not closed subset $C \subset X \times Y$. The projection on the factor X is then not closed and, by radially, some sequence of length $\lambda(C)$ converges outside of it. The main trick devised

for proving this theorem was to consider a subset C witnessing that $X \times Y$ is not pseudoradial and such that the length $\lambda(C)$ is minimal. From this a contradiction was reached.

However this proof presented a gap, and, probably, in order to overcome it, Bella and Gerlits were led to the following notion of a *semi-radial* space

Definition 2.7. A topological space is said to be *semi-radial* if for any not κ -closed subset $A \subset X$ there is a point $x \in \overline{A} \setminus A$ and a λ -sequence $(x_\alpha)_{\alpha < \lambda}$ in A such that $x_\alpha \rightarrow x$ and $\lambda \leq \kappa$.

We recall that a subset $A \subset X$ is said to be κ -closed if for all $B \subset A$, $|B| \leq \kappa$ implies $\overline{B} \subset A$.

The idea for fixing the gap in [12] was to consider a subset $A \subset X \times Y$ sequentially closed but not closed and so not κ -closed for some κ and to choose a counterexample having minimal κ . In fact Bella and Gerlits [6] proved more

Theorem 2.8. *The product of two compact Hausdorff pseudoradial spaces is pseudoradial if one of them is semi-radial.*

From this result one gets Th. 2.6 immediately as a corollary, once it is recognised that every radial space is semi-radial.

Let us recall that the *chain-character* of a pseudoradial space (see [2]), denoted by $\sigma_c(X)$, is the smallest cardinal number κ such that for any non-closed set $A \subset X$ there exist a point $x \in \overline{A} \setminus A$ and a sequence $(x_\alpha)_{\alpha < \lambda}$ which converges to x and satisfies $\lambda \leq \kappa$. In [5] a pseudoradial space was defined to be *R-monolithic* if for any $A \subset X$ the inequality $\sigma_c(\overline{A}) \leq |A|$ holds. It is easy to see that the following inclusions hold between the various classes of spaces.

Fréchet-Urysohn \subset Sequential \subset R-monolithic \subset Semi-radial \subset \subset Almost radial \subset Pseudoradial;

Fréchet-Urysohn \subset Radial \subset Semi-radial \subset Almost radial \subset Pseudoradial.

In general, all inclusions are proper and no relation can be established between radial and sequential or between radial and R-monolithic spaces. It is not known if (compact) Hausdorff almost radial spaces coincide with semi-radial ones.

The following theorem was proved in [5]

Theorem 2.9. *The product of countably many compact Hausdorff R -monolithic spaces is R -monolithic.*

In 1990 a striking result of B. Šapirovič [23] threw new and unexpected light on the class of pseudoradial spaces

Theorem 2.10. *Assuming the Continuum Hypothesis CH, a compact Hausdorff space X is sequentially compact if and only if it is pseudoradial.*

Observe that CH plays a role only in proving that a compact Hausdorff sequentially compact space is pseudoradial; the converse implication is easily proved without any additional set-theoretical assumption. As a consequence, we have that CH implies that the product of countably many compact Hausdorff pseudoradial spaces is pseudoradial. In fact, as is well known, sequential compactness is countably productive.

The proof of Šapirovič's result did not appear in written form for a long time. It appears now in the Proceedings of the Summer Conference in General Topology and its Applications (Madison, Wisconsin 1991) and was collected immediately after the conference by P. Nyikos and J. Vaughan, some months before the immature death of Boris Šapirovič [24]. However István Juhász and Zoltán Szentmiklóssy [18] were able to prove the following generalization of Boris result

Theorem 2.11. *If $2^\omega = \mathfrak{c} \leq \omega_2$ then every compact Hausdorff sequentially compact space is pseudoradial.*

3. New results on the product of pseudoradial spaces

We would like to point out the following interesting result given in [18]

Theorem 3.1. *Let X be a compact Hausdorff sequentially compact space (CSC for short hereafter) such that every non-empty closed subset $F \subset X$ has a point $p \in F$ with character $\chi(p, F) \leq \omega_1$. Then X is pseudoradial.*

Here, as usual, $\chi(p, X)$ denotes the least cardinality of a neighborhood base at a point $p \in X$. $\chi(X)$, the character of the space X , is the supremum for $p \in X$ of all $\chi(p, X)$ (see for example [11] or [16]).

As a consequence, taking into account the well known Čech - Pošpišil theorem (see for example [11] or [16]), one can conclude that every CSC space X such that $|X| < 2^{\omega_2}$ is pseudoradial.

The following result can then be proved

Theorem 3.2. *Let $(X_n)_{n \in \omega}$ be a countable family of compact Hausdorff*

pseudoradial spaces and for each $n \in \omega$ let $|X_n| < 2^{\omega_2}$. Then the cartesian product $\prod_{n \in \omega} X_n$ is pseudoradial.

Proof. Since all spaces are compact Hausdorff and pseudoradial, then they are also sequentially compact. Their countable product is then sequentially compact. However the cardinality of the product is strictly less than 2^{ω_2} . In fact, if $\kappa_n = |\prod_{i=0}^n X_i|$, then (κ_n) is an increasing sequence of cardinal numbers and each one is less than 2^{ω_2} . If $\kappa = \sup_{n \in \omega} \kappa_n \leq 2^{\omega_2}$ then $\text{cf}(\kappa) = \omega$; so it cannot be that $\kappa = 2^{\omega_2}$, since, as it is well known (see for example [19]) $\text{cf}(2^{\omega_2}) > \omega_2$ holds. Finally $\kappa = |\prod_{n \in \omega} X_n|$ is strictly less than 2^{ω_2} . So, by Th. 3.1, we can conclude that the compact, Hausdorff, sequentially compact space $\prod_{n \in \omega} X_n$ is pseudoradial. \diamond

The result given here is absolute, i.e. it holds in ZFC. A slight generalization of it can be proved but only consistently.

The following cardinal number was introduced in [7]

$$\mathfrak{h} = \min\{|\mathcal{F}| : (\forall F \in \mathcal{F})(F \text{ is nowhere dense in } \beta\omega \setminus \omega = \omega^*) \& (\cup \mathcal{F} \text{ is dense in } \omega^*)\}.$$

This is not the only way that \mathfrak{h} can be introduced, but is one of the simplest. It was shown in [13] that this cardinal number is the same as the μ considered in [9]; i.e. that it is the least cardinal number such that if \mathcal{F} , with $|\mathcal{F}| < \mu$, is a family of sequentially compact spaces, then the cartesian product of the family is again sequentially compact. So we have $\mathfrak{h} = \mu$. It is proved in [7] that $\omega_1 \leq \mathfrak{h} \leq \mathfrak{c} = 2^\omega$ and that \mathfrak{h} is a regular cardinal number. The following (consistently) more general result can be proved

Theorem 3.3. *Suppose that $\mathfrak{h} \leq \omega_3$ holds. Then, if \mathcal{F} is a family of strictly less than \mathfrak{h} compact Hausdorff pseudoradial spaces each one having cardinality $< 2^{\omega_2}$, the cartesian product $\prod \mathcal{F}$ is pseudoradial.*

Proof. In fact, the proof follows the same scheme as in the previous theorem. If κ is the cardinality of the product $\prod \mathcal{F}$, then $\text{cf} \kappa < \mathfrak{h}$, i.e. $\text{cf} \kappa \leq \omega_2$, whilst $\text{cf}(2^{\omega_2}) > \omega_2$. So $\kappa < 2^{\omega_2}$ and the product is still pseudoradial, by Th. 3.1. \diamond

Obviously if $\mathfrak{c} = \omega_1$ Th. 3.3 reduces to Th. 3.2.

So far, without restrictions on the cardinality or additional set-theoretical assumptions, we considered the product of one pseudoradial with a sequential or radial or semi-radial space (showing its pseudoradiality). Now we sharpen the result of Th. 2.8 showing

Theorem 3.4. *The product of two compact Hausdorff almost radial spaces is almost radial if one of them is semi-radial.*

Proof. Let us denote by $t\text{-Lim}A$ the set of limit points of $t\lambda$ -sequences in A ; obviously, in an almost radial space a subset A is closed if and only if $t\text{-Lim}A \subset A$. Let X be a compact T_2 semi-radial and Y a compact T_2 almost radial space and, by contradiction, suppose that $Z = X \times Y$ is not almost radial. Then a subset $A \subset Z$ exists such that A is not closed but $t\text{-Lim}A \subset A$, i.e. A is a non-closed set which is sequentially closed for t -sequences. Let κ be the least cardinal number such that A is not κ -closed, and let $B \subset A$, $|B| = \kappa$, $\overline{B} \setminus A \neq \emptyset$ and $(x, y) \in \overline{B} \setminus A$. The set $\{x\} \times Y$, homeomorphic to Y , is almost radial and $A \cap (\{x\} \times Y)$ is closed, since it contains all limits of t -sequences. Repeating an argument presented, for example, in [12], we can take a closed neighborhood V of (x, y) which does not intersect $A \cap (\{x\} \times Y)$. Substituting, if necessary, A with $A \cap V$ we can suppose that $x \notin \text{pr}_X(A)$. Then we have that $\text{pr}_X(A)$ is not κ -closed but it is $< \kappa$ -closed; in fact the X -projection is a closed map (Y is compact) and takes λ -closed sets into λ -closed sets. Since X is semi-radial there is a $t\kappa$ -sequence $(x_\alpha)_{\alpha < \kappa}$, whose elements are in $\text{pr}_X(A)$, $x_\alpha \rightarrow \bar{x} \notin \text{pr}_X(A)$. For any $\alpha \in \kappa$ chose an element y_α such that $(x_\alpha, y_\alpha) \in A$. It cannot be that κ many of the y_α coincide with some \tilde{y} : in this case a t -sequence (x_α, y_α) such that $y_\alpha = \tilde{y}$ would arise in A converging to $(\bar{x}, \tilde{y}) \notin A$, contrary to the assumption that A is sequentially closed. So there are κ distinct points in $\{y_\alpha\}$ and we can select a complete accumulation point p of this set. Since $(\bar{x}, p) \notin A$ we can assume as before that $p \notin \text{pr}_Y(A)$. For any $\alpha \in \kappa$, denote by C_α the closure in Y of the set $\{y_\beta : \beta < \alpha\}$ and let $C = \bigcup_{\alpha < \kappa} C_\alpha$. As $\text{pr}_Y(A)$ is $< \kappa$ -closed, $C \subset \text{pr}_Y(A)$, and it is not closed in Y since $p \in \overline{C} \setminus \text{pr}_Y(A)$. As Y is almost radial there is a t -sequence $(y'_\alpha)_{\alpha < \kappa}$ converging to a point $\bar{y} \notin C$. By taking, if necessary, a subsequence, we can assume that $y'_\alpha \notin C_\alpha$ for any $\alpha < \kappa$. Fix a function $f: \kappa \rightarrow \kappa$ in such a way that $y'_\alpha \in \{y_\beta : \alpha < \beta < f(\alpha)\}$ and select a point $x'_\alpha \in \{x_\beta : \alpha < \beta < f(\alpha)\}$ such that $(x'_\alpha, y'_\alpha) \in A$ for all $\alpha < \kappa$; this is possible since A is $< \kappa$ -closed. This sequence $(x'_\alpha, y'_\alpha) \rightarrow (\bar{x}, \bar{y}) \notin A$ and is a t -sequence since A is $< \kappa$ -closed. Contradicting that A is closed for t -sequences. \diamond

Corollary 3.5. *The product of a finite number of compact Hausdorff semi-radial spaces is almost radial.*

If it could be proved that compact Hausdorff almost radial spaces coincide with the semi-radial ones, this result would show that the class

of compact almost radial spaces is stable under finite products. Let us recall one more result from [18].

Theorem 3.6. *Let X be a compact Hausdorff space which is not pseudoradial. Then there is a subspace $Y \subset X$ satisfying $|Y| < \mathfrak{c}$, more precisely, $|Y| \leq (\mathfrak{c}^-)^-$ such that \bar{Y} is not pseudoradial.*

Here, for a cardinal κ , κ^- denotes the predecessor of it; i.e. $\kappa^- = \lambda$ if $\kappa = \lambda^+$, $\kappa^- = \kappa$ if κ is a limit cardinal. Taking into account this and Th. 2.9 one gets immediately the following

Theorem 3.7. *Let $(X_i)_{i \in \omega}$ be a countable family of compact Hausdorff pseudoradial spaces such that every closed subspace whose density is less than \mathfrak{c} is R -monolithic. Then $\prod_{i \in \omega} X_i$ is pseudoradial.*

Here, by the *density* of a space X we mean the least cardinal number, denoted by $d(X)$, of a subset $D \subset X$ whose closure gives X .

Finally, for the sake of completeness, we would like to mention one more result due to one of the authors which concerns the non-compact case and generalizes Th. 2.3; this result is still unpublished (see [8]).

Remember that an Hausdorff space X is said to be $[\lambda, \lambda]$ -compact for a regular λ (see [3], p. 17) if one of the following equivalent properties hold: (i) Every subset M of regular cardinality $|M| = \lambda$ has a complete accumulation point. (ii) Every well ordered decreasing sequence of nonempty closed sets $(F_\alpha)_{\alpha < \lambda}$ has a nonempty intersection. (iii) Every open cover \mathcal{U} , with $|\mathcal{U}| = \lambda$, has a subcover \mathcal{V} having cardinality $|\mathcal{V}| < \lambda$. A topological space is called a λ -chain-net space if every non-closed subset of it contains a convergent λ -sequence.

Then the following was proved

Theorem 3.8. *Let X be a chain-net Hausdorff $[\lambda, \lambda]$ -compact topological space with no convergent (non-trivial) sequence of length less than λ , and Y a λ -chain-net space with λ regular. Then $X \times Y$ is chain-net. As it is easy to see, this theorem reduces to Th. 2.3 if $\lambda = \omega$.*

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