Mathematica Pannonica
5/2 (1994), 271 – 273

SMALL n-DOMINATING SETS

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Received August 1994

AMS Subject Classification: 05 C 35

Keywords: Graph, domination, distance domination, total distance domination

Abstract: We prove the inequality $\gamma_n(G) + (n-1)\gamma_n^t(G) \leq p$, where G is any connected graph on $p \geq 2n-1$ vertices, and $\gamma_n(G)$ and $\gamma_n^t(G)$ denote the minimum cardinalities of vertex sets D and D_t such that each vertex x is at distance less than n from some $y \in D_t$, $y \neq x$, and each $x \notin D$ is at distance less than n from some $y \in D$. Our method yields a very short proof of a recent theorem due to Henning et al. [Math. Pannon. **5**/1 (1994), 67–77].

In this note we provide a very short proof of a recent result due to Henning et al. [1] on generalized domination parameters of graphs. Our method, at the same time, also yields a somewhat stronger assertion. (For further related results, see [2].)

Let G be a *connected* graph with a p-element vertex set $V(G)$ $(p > 1)$ and with edge set $E(G)$. The *distance* $\mathrm{d}(x,y)$ of two vertices $x, y \in V(G)$ is the smallest number of edges in a path joining x to y. Let $n > 1$ be an integer. Adopting the terminology of [1], a $D \subseteq V(G)$ is a $P_{\leq n}$*-dominating set* (*total* $P_{\leq n}$*-dominating set*, respectively) if each vertex $x \in V(G) - D$ (each $x \in V(G)$) is at distance less than n from some $y \in D$, $y \neq x$. (Sometimes such a D is simply called a (total) $(n-1)$-dominating set in the literature.) The minimum cardinality of a $P_{\leq n}$-dominating set and of a total $P_{\leq n}$-dominating set is denoted by $\gamma_n(G)$ and $\gamma_n^t(G)$, respectively.

The main result of [1] states

$$\gamma_n(G) + \gamma_n^t(G) \le 2p/n \tag{1}$$

provided that G is connected and its order, p, is at least $2n$. Here we prove the following stronger assertion.

Theorem 1. *If G is a connected graph of order $p \ge 2n-1$, $n \ge 2$, then*

$$\gamma_n(G) + (n-1)\gamma_n^t(G) \le p\,. \tag{2}$$

Proof. Since every (total) $P_{\le n}$-dominating set of any spanning tree T of G is a (total) $P_{\le n}$-dominating set in G as well, it suffices to prove the assertion for trees. Hence, let T be a tree of order $p \ge 2n-1 \ge$ ≥ 3. We denote $\mathrm{d}(T) = \max_{x,y\in V(T)} \mathrm{d}(x,y)$ (the *diameter* of T) and $r(T) = \min_{x\in V(T)} \max_{y\in V(T)} \mathrm{d}(x,y)$ (the *radius*).

Suppose first that $\mathrm{d}(T) \le 2n-2$. Then $r(T) \le n-1$, and a 'central' vertex within distance $n-1$ from every vertex of T forms a $P_{\le n}$-dominating set. Therefore, $\gamma_n(T) = 1$ and $\gamma_n^t(T) = 2$, i.e., $\gamma_n(T) + (n-1)\gamma_n^t(T) = 2n-1 \le p$. Hence, the assertion is valid for 'small' diameter (which is always the case for $p = 2n-1$), allowing us to apply induction on p. This can be done if T has an edge e such that both components T_1, T_2 of $T-e$ contain at least $2n-1$ vertices (and, in particular, whenever $\mathrm{d}(T) \ge 4n-3$). Indeed, in this case $\gamma_n(T) \le \gamma_n(T_1) + \gamma_n(T_2)$ and $\gamma_n^t(T) \le \gamma_n^t(T_1) + \gamma_n^t(T_2)$, thus $\gamma_n(T) +$ $+(n-1)\gamma_n^t(T) \le (\gamma_n(T_1) + (n-1)\gamma_n^t(T_1)) + (\gamma_n(T_2) + (n-1)\gamma_n^t(T_2)) \le$ $\le |V(T_1)| + |V(T_2)| = p$ follows by induction.

Suppose $2n-1 \le d(T) \le 4n-4$, and that $T-e$ contains a 'small' component for each edge $e \in E(T)$. Choose an edge $e = uv$ such that the smaller component (the one of order $\le 2n-2$), say the component containing u, is as large as possible. Then *all* components $T_1, T_2, \ldots, T_m$ of $T - v$ have orders at most $2n-2$. Define the height h_i of T_i as the length of a longest path $P_i \subseteq \langle T_i \cup v\rangle$ starting at v, where $\langle T_i \cup v\rangle$ denotes the subgraph induced by $V(T_i) \cup \{v\}$. By the assumption $\mathrm{d}(T) \ge 2n-1$, some T_i have $h_i \ge n$; say, $h_i \ge n$ for $1 \le$ $\le i \le k$ and $h_i < n$ for $k < i \le m$ (where $k = m$ is possible). For $i \le k$ we denote by v_i the vertex of P_i at distance $h_i - n + 1$ from v. Since $|V(T_i)| \le 2n-2$, v_i $P_{\le n}$-dominates T_i for each i, moreover v $P_{\le n}$-dominates $\{v_1, \ldots, v_k\} \cup \{T_j \mid k < j \le m\}$. Hence, $\gamma_n(T) \le$ $\le \gamma_n^t(T) \le k+1$, implying (2) for $p \ge (k+1)n$.

To obtain sharper bounds on $\gamma_n(T)$ and $\gamma_n^t(T)$, we consider the subtree $T' = \langle T'' \cup \{v_1, \ldots, v_k\}\rangle$, where T'' is the connected component of $T - \{v_1, \ldots, v_k\}$ containing v. If $\mathrm{d}(T'') < n - 3$, we have $\gamma_n(T) = \gamma_n^t(T) = k$ and $p > |V(T_1)| + \ldots + |V(T_k)| \geq kn$, implying $\gamma_n(T) + (n-1)\gamma_n^t(T) < p$. On the other hand, if $\mathrm{d}(T'') \geq n - 2$, then $p \geq (k+1)n - 1$ holds, with equality if and only if T'' has order $n - 1$ and the subtree rooted at v_i is a path of length $n - 1$ in each T_i. In this case, however, $\{v_1, \ldots, v_k\}$ is already a dominating set, therefore $\gamma_n(T) = k$, $\gamma_n^t(T) \leq k + 1$, $\gamma_n(T) + (n-1)\gamma_n^t(T) \leq (k+1)n - 1 = p$. $\Diamond$

Remarks. Since $\gamma_n(G) \leq \gamma_n^t(G)$ whenever G has no isolated vertices, the inequality (2) immediately implies (1). Certainly, every example showing the tightness of (1) (see [1] where an infinite family of graphs G with $\gamma_n(G) = \gamma_n^t(G) = p/n$ is exhibited) yields that (2) is tight, too. However, (2) is best possible in a much stronger sense as well; namely, its left-hand side cannot be replaced by $(1-\varepsilon)\gamma_n(G) + (n - 1 + \varepsilon)\gamma_n^t(G)$, for any $\varepsilon > 0$. To see this, take $k - 1$ (≥ 1) vertex-disjoint paths $T_1, \ldots, T_{k-1}$ of length $n-1$ and one path of length $2n-2$. Joining a new vertex v with one endpoint of each T_i, we obtain a tree T of order $p = (k+1)n - 1$, with $\gamma_n(T) = k$ and $\gamma_n^t(T) = k + 1$, hence $\gamma_n(T) + (n-1)\gamma_n^t(T) = p$ and $\gamma_n(T) < \gamma_n^t(T)$. Further 'isolated' examples are the paths on $3n - 1$, $4n - 2$, $4n - 1$ vertices (the corresponding parameters are $\gamma_n = 2, 2, 3$ and $\gamma_n^t = 3, 4, 4$), and all connected graphs G of order $p = 2n - 1$ ($\gamma_n(G) = 1$ and $\gamma_n^t(G) = 2$). It may be true, however, that if p is 'sufficiently large' with respect to n, then $\gamma_n(G) + (n-1)\gamma_n^t(G) < p$ holds unless $\gamma_n(G) = \gamma_n^t(G) = p/n$, or G is a k-branched tree constructed above plus possibly a few additional edges among its 'short' branches.

References

[1] HENNING, M. A., OELLERMANN, O. R. and SWART, H. C.: Relations between distance domination parameters, *Mathematica Pannonica* **5**/1 (1994), 67–77.

[2] HENNING, M. A., OELLERMANN, O. R. and SWART, H. C.: Relating pairs of distance domination parameter, *J. Comb. Math. Comb.* (to appear).