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SOME PROBLEMS IN NUMBER THEORY, COMBINATORICS AND COMBINATORIAL GEOMETRY

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Abstract: Some open problems are posed on topics as given in the title. The solution of some of the problems will be awarded in U.S.$ by the author.

1. Number theory

1.1. In a forthcoming paper of A. Sárközy, V. T. Sós and myself we investigate the following problem: Denote by $F_k(n)$ the size of the largest set of integers $1 \leq a_1 < a_2 < \cdots < a_l \leq n$, $l = F_k(n)$ for which the product of no k a's can be a square. We obtain fairly accurate upper and lower bounds for $F_k(n)$.

We also discussed the following problem which is not mentioned in our paper: Let $1 \leq a_1 < a_2 \ldots$ be an infinite sequence of integers, assume that the product of an odd number of a's is never a square. I. Ruzsa proved that the density of such a sequence is at most $1/2$ (Th. 4.1 in the cited paper). Clearly it can be $1/2$. To see this let the a's have an odd number of distinct prime factors.

A related problem states as follows: Let $a_1 < a_2 < \cdots < a_l \leq n$, assume that the product of an odd number of a's is never a square. Denote $\max l = g(n)$. *Determine or estimate $g(n)$ as accurately as possible.* It is easy to see that for a fixed but small $c > 0$, $g(n) > n\left(\frac{1}{2} + c\right)$, and I. Ruzsa showed that $g(n) < n(1-c)$ (Th. 4.2 in the cited paper).

ERDŐS, P., SÁRKÖZY, A. and SÓS, V. T.: On product representation of integers I. The paper will soon appear in the *European Journal of Combinatorics*
RUZSA, I.: General multiplicative functions, *Acta Arithm.* **32** (1977), 313–347.

1.2. This is a problem of P. Cameron and myself. Denote by $f(n)$ the number of sequences of integers $1 \leq a_1 < a_2 < \cdots < a_t \leq n$ (t is not fixed) for which no a_i is the distinct sum of other a's. *Is it true that*

$$f(n) = 2^{\frac{n}{2}(1+o(1))} \ ? \tag{1}$$

The integers $\frac{n}{2} \leq t \leq n$ show that $f(n) > 2^{\frac{n}{2}}$, but we expect that $f(n)$ is not very much larger than $2^{\frac{n}{2}}$.

CAMBERON, P. and ERDŐS, P.: On the number of sets of integers with various properties, Number Theory, Banff (A.B. 1988) de Gruyter, Berlin, 1990, 61–79.

1.3. Let $A = \{a_1 < a_2 < \dots\}$ be an infinite sequence of integers. $f(n)$ denotes the number of solutions of $n = a_i + a_j$. $A + A$ will denote the set of integers which can be written in the form $a_i + a_j$ i.e. the set of integers for which $f(n) > 0$.

A is called a basis of order r if every integer is the sum of r or fewer a's. It is called an asymptotic basis of order r if all but a finite number of integers are the sum of r or fewer a's. An old conjecture of P. Turán and myself states that *if A is an asymptotic basis of order 2 then*

$$\overline{\lim} f(n) = \infty \tag{1}$$

and perhaps there is an $\varepsilon > 0$ for which for infinitely many n

$$f(n) > \varepsilon \log n. \tag{2}$$

I offer for a proof or disproof of (1) 500 dollars.

Perhaps (1) *holds already if we only assume that the upper density of the integers for which $f(n) > 0$ is* 1.

It follows from an old result of mine that (2) if true is best possible (apart from the value of ε).

Unfortunately these old problems seem unattackable at the moment. Perhaps the following related problem is not hopeless. Denote by

$g_A(x)$ the number of integers $n < x$ for which $f(n) > 0$ $\Big($i.e. $g_A(x) = \sum_{\substack{n<x \\ n=a_i+a_j}} 1,\ a_i, a_j \in A\Big)$. *Is it true that for every $\varepsilon > 0$ there is a sequence A for which for every x*

$$g(x) > (1-\varepsilon)x$$

but for every n, $f(n) < c(\varepsilon)$. In other words $f(n)$ is bounded but most of the numbers can be written in the form $a_i + a_j$. I would be satisfied if one would show *that there is a sequence A for which the upper density of the integers n with $f(n) > 0$ is $> 1 - \varepsilon$ but $f(n) < c(\varepsilon)$.* Both A. Sárközy and I believe that *if $f(n) < c$ then the upper density of the integers n with $f(n) > 0$ is $< 1 - \varepsilon$ where $\varepsilon = \varepsilon(c)$.*

For the literature see the excellent book of HALBERSTAM, H. and ROTH, K. F.: Sequences, Springer Vrlg, Berlin.

1.4. M. Nathanson, J. Spencer and I proved a few years ago that there is a basis of order three for which $f(n) \le 2$ with at most finitely many exceptions. We used the probability method. *Perhaps there is a basis of order three for which $f(n) = 1$ for all but finitely many exceptions* (i.e. there is a sequence A for which every integer $n = a_i + a_j + a_k$ but the integers $a_M + a_N$ are all distinct with a possible finite number of exceptions). It is not very likely that the probability method will help here.

1.5. St. Burr and I posed a few years ago the following problem. Let A be a sequence of integers for which the density of the integers with $f(n) > 0$ is positive. *Can one always decompose the sequence A as the union of two disjoint subsequences $A = A_1 \cup A_2$ for which the density of $A_1 + A_1$ and $A_2 + A_2$ is also positive?* As far as I remember we could not settle this question. While writing this paper I several times thought that I can prove that *there is a basis A of order two for which for any decomposition $A = A_1 \cup A_2$ the sequences $A_1 + A_1$ and $A_2 + A_2$ can not both have bounded gaps.* But unfortunately I could never quite finish the proof. Thus the problem is still open.

For problems 1.3, 1.4 and 1.5 the interested reader should consult besides the book of H. Halberstam and K. F. Roth also several recent papers of M. Nathanson, J. Spencer, P. Tetali and myself, many of which are joint papers.

1.6. A sequence $A = \{a_1 < a_2 < \cdots < a_l \leq n\}$ is called a Sidon sequence if the sums $a_i + a_j$ are all distinct. Put $\max l = g(n)$. P. Turán and I proved

$$(1) \qquad g(n) < n^{\frac{1}{2}} + cn^{\frac{1}{4}}.$$

Perhaps

$$(2) \qquad g(n) = n^{\frac{1}{2}} + O(1).$$

(2) is perhaps too optimistic, but I am fairly sure that *for every* $\varepsilon > 0$

$$(3) \qquad g(n) = n^{\frac{1}{2}} + o(n^{\varepsilon}).$$

I offer 500 dollars for a proof or disproof of (3).

I conjecture that *for every* t *and* $n > n_0(t)$

$$(4) \qquad g(n+t) \leq g(n) + 1$$

and *perhaps for* $t < \varepsilon n^{\frac{1}{2}}$

$$(5) \qquad g(n+t) \leq g(n) + 1.$$

(4) and (5) if true would imply that the growth of $g(n)$ is fairly regular.

The older literature on Sidon sequences can be found in the book of H. Halberstam and K. F. Roth "Sequences".

2. Combinatorics

2.1. Let $G(n)$ be a graph of n vertices. Assume that there is a k for which every subgraph of m vertices $(1 \leq m \leq n)$ of our $G(n)$ has an independent set of size $\frac{m}{2} - k$ (k is fixed, n is arbitrary). Is it then true that *the vertex set of* G *can be decomposed into three disjoint sets* $S_1 \cup S_2 \cup S_3$ where S_1 and S_2 *are independent and* $S_3 < f(k)$, *i.e.* $G(n)$ *is the union of a bipartite graph and a bounded set*? The question is open even for $k = 1$.

A. Hajnal and I proved that there is a graph G of infinite chromatic number every subgraph of m vertices of which contains an independent set of size $\frac{m}{2}(1-\varepsilon)$. *Does this remain true if* $\frac{m}{2}(1-\varepsilon)$ *is replaced by* $\frac{m}{2} - f(m)$ *where* $f(m) \to \infty$?

A. Hajnal, E. Szemerédi and I have the following very annoying unsolved problem: Let $h(n)$ tend to infinity arbitrarily slowly. Is it true that *there is a G of infinite chromatic number every subgraph of n vertices of which can be made bipartite by the omission of $\leq h(n)$ edges*? I offer 250 dollars for a proof or disproof.

ERDŐS, P., HAJNAL, A. and SZEMERÉDI, E.: On almost bipartite large chromatic graphs, *Annals of Discrete Math.* **12** (1982), 117–123.

2.2. Denote by $G(n; f(n))$ a graph of n vertices and $f(n)$ edges. It is easy to see that every $G(2n+1; 3n+1)$ contains an even cycle and $3n+1$ is best possible.

Let now $\mathcal{S}$ be a subsequence of the even numbers. Let $f(n;\mathcal{S})$ be the smallest integer for which every $G(n; f(n;\mathcal{S}))$ contains a cycle whose length is in $\mathcal{S}$. In particular *can we find a sequence $\mathcal{S}$ of density 0 for which $f(n;\mathcal{S}) < cn$?* A. Gyárfás and I conjectured that if the sequence $\mathcal{S}$ consists of the powers of 2 then $f(n;\mathcal{S})/n \to \infty$. We have no guess *what happens if $\mathcal{S}$ consists of the numbers $2u^2$*. It follows from an old result of B. Bollobás that if $\mathcal{S}$ is an arithmetic progression of even numbers then $f(n;\mathcal{S}) < cn$.

BOLLOBÁS, B.: Cycles modulo *k*, *Bull. London Math. Soc.* **9** (1977), 97–98.

2.3. Problem 2.2 originated as follows. A. Hajnal and I conjectured that *if G has infinite chromatic number and if $n_1 < n_2 < \dots$ is the sequence of the sizes of distinct odd cycles of G then $\sum \frac{1}{n_i} = \infty$ and perhaps $\sum_{n_i < x} 1 > cx$ holds for infinitely many x.* We never could get anywhere with this problem. P. Mihok and I later conjectured that *if G has infinite chromatic number then for infinitely many u G has a cycle of length 2^u*. This problem also remained unattackable.

Three years ago A. Gyárfás and I conjectured that if G is a graph every vertex of which has degree ≥ 3 then G has a cycle of length 2^u. We finally thought that this was not true but could not find a counterexample, we concluded that *probably for every k there is a graph every vertex of which has degree $\geq k$ but there is no cycle of length 2^u.* The problem is still open and perhaps is not very difficult.

2.4. Here is a problem of R. Faudree and myself. Consider all the graphs of n vertices. Denote by

$$(1) \qquad 3 \le a_1 < a_2 < \cdots < a_t \le n$$

the lengths of the cycles occurring in any of these graphs. Denote by $f(n)$ the number of possible sequences (1). Clearly $f(n) \le 2^{n-2}$. We easily showed $f(n) > 2^{\frac{n}{2}}$. *Probably*

$$f(n)^{\frac{1}{n}} \to c, \qquad \sqrt{2} \le c < 2.$$

It is easy to see that for $n \ge 5$, $f(n) < 2^{n-2}$ but we could not prove *that $f(n)/2^n \to 0$ and that $f(n)/2^{\frac{n}{2}} \to \infty$.*

2.5. Let $f(n)$ be the smallest integer for which every graph of n vertices every vertex of which has degree $\ge f(n)$ contains a C_4 (i.e. a cycle of length 4). *Is it true that for $n > n_0$*

$$(1) \qquad f(n+1) \ge f(n)?$$

If this is too optimistic is it at least true that *there is an absolute constant c for which for every $m > n$*

$$(2) \qquad f(m) > f(n) - c?$$

The proof of (2) is perhaps easy, but so far the problem is open.

2.6. Let G be a four-chromatic graph, $m_1 < m_2 < \ldots$ be the lengths of the cycles of G, can $\min(m_{i+1} - m_i)$ *be arbitrarily large? Can this happen if the girth of G is large?*

2.7. R. Faudree, R. H. Schelp and I have the following question: Let $G(n)$ have girth $> 2s$ and every vertex has degree $\ge k$. Is it then true that *the number of cycles of distinct lengths of our $G(n)$ is $> ck^s$?* We proved this conjecture only for $s = 2$ and ran into unexpected difficulties already for $s = 3$.

2.8. The n-dimensional cube $C^{(n)}$ has 2^n vertices and $n2^{n-1}$ edges. A very old conjecture of mine states that *every subgraph of* $(1+\varepsilon)n2^{n-2}$

edges of $C^{(n)}$ contains a C_4 (i.e. a $C^{(2)}$). This conjecture is still open and I offer 100 dollars for a proof or disproof. Denote by $f(n)$ the smallest integer for which every subgraph of $C^{(n)}$ of $f(n)$ edges contains a C_4. I conjectured that

$$f(n) < n2^{n-2} + C2^n$$

for sufficiently large C, but here I overconjectured since H. Harborth and his students proved

$$f(n) > n2^{n-2} + n^\alpha 2^n$$

for some fixed positive α. *The exact determination of $f(n)$ is perhaps not hopeless and would even be of some interest for small values of n. Is there an Erdős–Stone type theorem for the subgraphs of $C^{(n)}$?*

ERDŐS, P. and STONE, A.: On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1940), 1087–1091.
CHUNG, F. K. R.: Subgraphs of a hypercube containing no small even cycles, *J. Graph Theory* **16** (1992), 273–286.
CONDER, M.: Hexagon free subgraphs of hypercubes, *J. Graph Theory* **17** (1993), 477–479.

3. Combinatorial geometry

3.1. Let $f_k(n)$ be the largest integer for which there are n points $x_1, \dots, x_n$ in k-dimensional Euclidean space for which for every x_i there are at least $f_k(n)$ points x_j equidistant from x_i. First let us discuss $k = 2$ i.e. our points are in the plane. I conjectured *$f_2(n) < n^\varepsilon$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$, and perhaps $f_2(n) < n^{c/\log\log n}$*. The lattice points show that $f_2(n) > n^{c_1/\log\log n}$.

In 1946 I conjectured that *among any n points in the plane the same distance can occur at most $n^{1+c/\log\log n}$ times* and the above conjectures would give a very considerable strengthening. I offer 500 dollars for a proof of $f_2(n) = o(n^\varepsilon)$, but only 50 dollars for a counterexample. $f_2(n) < cn^{\frac{1}{2}}$ is trivial and J. Pach points out that a result of J. Pach and M. Sharir gives $f_2(n) < cn^{2/5}$. Any further improvement would be very welcome. A recent letter of P. C. Fishburn shows that perhaps the set $x_1, \dots, x_n$ which gives the largest value of $f_2(n)$ may not be given by the lattice points. Fishburn proved that 6 is the smallest integer for which $f_2(6) = 3$ and 8 the smallest integer for which $f_2(8) = 4$.

E. Makai, J. Pach and I proved

$$c_1 n^{\frac{1}{3}} < f_3(n) < c_2 n^{\frac{3}{4}} \tag{1}$$

and

$$\frac{n}{2} + 2 \le f_4(n) < \frac{n}{2}(1 + o(1)). \tag{2}$$

In (1) the upper bound holds in the stronger form that if among our n points $x_1, \ldots, x_n$ there are $k = c\sqrt{n}$ points, say $y_1, \ldots, y_k$, no three of these on a line, then there exists an i, $1 \le i \le k$, such that y_i does not have more than $c_2 n^{3/4}$ equidistant points x_j from it. *Probably*

$$f_4(n) = \frac{n}{2} + O(1).$$

In a finite time we may publish a quadruple paper about these results.

3.2. Denote by $\mathrm{d}(x, y)$ the distance between the points x and y and by $\mathrm{D}(x_1, \ldots, x_n)$ denote the diameter i.e. the maximum of $\mathrm{d}(x_i, x_j)$. Let $x_1, x_2, \ldots, x_n$ be n points in the plane for which $\mathrm{d}(x_i, x_j) \ge 1$ and the diameter is minimal. It has been known since A. Thue (1910) that asymptotically the minimum is given by the triangular lattice. Let $x_1, \ldots, x_n$ be a set which implements the minimum of the diameter. Denote by $h(n)$ the number of incongruent sets which implement the minimum of the diameter. I would guess that $h(n) \to \infty$ *as* $n \to \infty$ but as far as I know it has not even been proved that $h(n) \ge 2$ *for* $n > n_0$. It is generally believed that *for* $n > n_0$ *no subset of the triangular lattice implements the minimum of the diameter but that any set* $x_1, \ldots, x_n$ *which implements the minimum of the diameter has a large intersection with the triangular lattice*, but as far as I know nothing has been proved.

I conjectured that *if* $x_1, \ldots, x_n$ *implements the minimum of the diameter and if* $n > n_0$ *then our set contains an equilateral triangle of sides* 1. I could not prove it but felt that it should not be hard. To my great surprise both B. H. Sendov and M. Simonovits doubted the truth of this conjecture. I offer 100 dollars for a counterexample for infinitely many values of n but only 50 dollars for a proof.

3.3. Inscribe n non-overlapping squares into a unit square. Denote by $a_1, a_2, \ldots, a_n$ the sides of these squares. Let

$$f(n) = \max \sum_{i=1}^{n} a_i.$$

It is easy to see that

$$f(k^2) = k$$

and *I conjectured that*

$$f(k^2 + 1) = k. \tag{1}$$

I conjectured (1) more than 60 years ago. Perhaps the proof (or disproof) of (1) will not be difficult.

3.4. Let $x_1, x_2, \ldots, x_n$ be n points in the plane in general position i.e. no three on a line and no four on a circle. Let $h(n)$ be the largest integer for which these points determine at least $h(n)$ distinct distances. I conjectured *that* $h(n)/n \to \infty$ but could not even prove *that* $h(n) > n - 1$, in fact I could not exclude *that* $h(n) < cn$ *for some* $c < 1$.

3.5. It is easy to see by the well-known construction of H. Lenz that one can give $3n$ points in six-dimensional space which determine $n^3 + 6n^2$ equilateral triangles of size 1 for $4 \mid n$. It suffices to take three suitable orthogonal circles and take n points on each of them which form $\frac{n}{4}$ inscribed squares. I conjectured that *in six-dimensional space one cannot have* $3n$ *points which determine* $(1 + \varepsilon)n^3$ *equilateral triangles of size* 1. If one just asks *for equilateral triangles of any size* one can of course get somewhat more equilateral triangles, but *their maximal number is probably less than* $(1+\varepsilon)n^3$. Perhaps I overlook a trivial point. Many related questions can be asked but I leave their formulation to the interested reader.