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MEROMORPHIC STARLIKE UNIVALENT FUNCTIONS WITH ALTERNATING COEFFICIENTS

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Abstract: Coefficient estimates and distortion theorems are obtained for meromorphic starlike univalent functions with alternating coefficients. Further class preserving integral operators are obtained.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \quad = \frac{(z^2 f(z))'}{z}.$$

$$D^2 f(z) = D(D^1 f(z)).$$

and for $n = 1, 2, 3, \dots$

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m = \frac{(zD^{n-1} f(z))'}{z}.$$

In [4] Uralegaddi and Somanatha obtained a new criteria for meromorphic starlike univalent functions via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha)$, $0 \leqq \alpha < 1$, $n \in \mathbb{N}_0 = \{0, 1, \dots\}$, where $B_n(\alpha)$ is the class consisting of functions in $\sum$ satisfying

$$\text{(1.2)} \quad \operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 2 \right\} < -\alpha, \quad |z| < 1, \quad 0 \leqq \alpha < 1, \quad n \in \mathbb{N}_0.$$

The condition (1.2) is equivalent to

$$\text{(1.3)} \qquad \frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + (3 - 2\alpha) w(z)}{1 + w(z)},$$

$w(z) \in H = \{w \text{ regular, } w(0) = 0 \text{ and } |w(z)| < 1, z \in U = \{z \colon |z| < 1\}\}$, or, equivalently,

$$\text{(1.4)} \qquad \left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{\frac{D^{n+1} f(z)}{D^n f(z)} + 2\alpha - 3} \right| < 1.$$

We note that $B_0(\alpha) = \Sigma^*(\alpha)$, is the class of meromorphically starlike functions of order α $(0 \leq \alpha < 1)$ and $B_0(0) = \Sigma^*$, is the class of meromorphically starlike functions.

Let σ_A be the subclass of Σ which consists of functions of the form

$$\text{(1.5)} \quad f(z) = \frac{1}{z} + a_1 z - a_2 z^2 + a_3 z^3 \dots = \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^{m-1} a_m z^m, \quad a_m \geq 0$$

and let $\sigma^*_{A,n}(\alpha) = B_n(\alpha) \cap \sigma_A$.

In this paper coefficient inequalities, distortion theorems for the class $\sigma^*_{A,n}(\alpha)$ are determined. Techniques used are similar to these of Silverman [2] and Uralegaddi and Ganigi [3]. Finally, the class preserving integral operators of the form

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \qquad (c > 0) \tag{1.6}$$

is considered.

2. Coefficient inequalities

Theorem 1. *Let* $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$. *If*

$$\sum_{m=1}^{\infty} (m+2)^n (m+\alpha)|a_m| \le (1-\alpha), \tag{2.1}$$

then $f(z) \in B_n(\alpha)$.

Proof. Suppose (2.1) holds for all admissible values of α and n. It suffices to show that

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{\frac{D^{n+1} f(z)}{D^n f(z)} + 2\alpha - 3} \right| < 1 \quad \text{for} \quad |z| < 1.$$

We have

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{\frac{D^{n+1} f(z)}{D^n f(z)} + 2\alpha - 3} \right| = \left| \frac{\sum_{m=1}^{\infty} (m+2)^n (m+1) a_m z^{m+1}}{2(1-\alpha) - \sum_{m=1}^{\infty} (m+2)^n (m-1+2\alpha) a_m z^{m+1}} \right| \le$$

$$\le \frac{\sum_{m=1}^{\infty} (m+2)^n (m+1)|a_m|}{2(1-\alpha) - \sum_{m=1}^{\infty} (m+2)^n (m-1+2\alpha)|a_m|}.$$

The last expression is bounded above by 1, provided

$$\sum_{m=1}^{\infty} (m+2)^n (m+1)|a_m| \le 2(1-\alpha) - \sum_{m=1}^{\infty} (m+2)^n (m-1+2\alpha)|a_m|$$

which is equivalent to (2.1), and this is true by hypothesis. ◊

For functions in $\sigma^*_{A,n}(\alpha)$ the converse of the above theorem is also true.

Theorem 2. *A function $f(z)$ in σ_A is in $\sigma^*_{A,n}(\alpha)$ if and only if*

$$\sum_{m=1}^{\infty}(m+2)^n(m+\alpha)a_m \leq (1-\alpha). \tag{2.2}$$

Proof. In view of Th. 1 it suffices to show the only if part. Suppose

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-2\right\}= \tag{2.3}$$
$$=\operatorname{Re}\left\{\frac{-\frac{1}{z}+\sum\limits_{m=1}^{\infty}(-1)^{m-1}(m+1)^n m a_m z^m}{\frac{1}{z}+\sum\limits_{m=1}^{\infty}(-1)^{m-1}(m+2)^n a_m z^m}\right\} < -\alpha.$$

Choose values of z on the real axis so that $\left(\frac{D^{n+1}f(z)}{D^n f(z)}-2\right)$ is real. Upon clearing the denominator in (2.3) and letting $z \to -1$ through real values, we obtain

$$1-\sum_{m=1}^{\infty}(m+2)^n m a_m \geq \alpha\Big(1+\sum_{m=1}^{\infty}(m+2)^n a_m\Big)$$

which is equivalent to (2.2). ◊

Corollary 1. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then*

$$a_m \leq \frac{(1-\alpha)}{(m+2)^n(m+\alpha)} \qquad (m \geq 1).$$

Equality holds for the functions of the form

$$f_m(z)=\frac{1}{z}+(-1)^{m-1}\frac{(1-\alpha)}{(m+2)^n(m+\alpha)}z^m.$$

3. Distortion theorems

Theorem 3. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{1-\alpha}{3^n(1+\alpha)}r \leq |f(z)| \leq \frac{1}{r}+\frac{1-\alpha}{3^n(1+\alpha)}r \tag{3.1}$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{1-\alpha}{3^n(1+\alpha)}z \quad at\ z=r, ir. \tag{3.2}$$

Proof. Suppose $f(z)$ is in $\sigma^*_{A,n}(\alpha)$. In view of Th. 2, we have

$$3^n(1+\alpha)\sum_{m=1}^{\infty}a_m \le \sum_{m=1}^{\infty}(m+2)^n(m+\alpha)a_m \le (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty}a_m \le \frac{1-\alpha}{3^n(1+\alpha)}.$$

Consequently, we obtain

$$|f(z)| \le \frac{1}{r}+\sum_{m=1}^{\infty}a_m r^m \le \frac{1}{r}+r\sum_{m=1}^{\infty}a_m \le \frac{1}{r}+\frac{1-\alpha}{3^n(1+\alpha)}r.$$

Also

$$|f(z)| \ge \frac{1}{r}-\sum_{m=1}^{\infty}a_m r^m \ge \frac{1}{r}-r\sum_{m=1}^{\infty}a_m \ge \frac{1}{r}-\frac{1-\alpha}{3^n(1+\alpha)}r.$$

Hence the results (3.1) follow. ◊

Theorem 4. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r^2}-\frac{1-\alpha}{3^n(1+\alpha)} \le |f'(z)| \le \frac{1}{r^2}+\frac{1-\alpha}{3^n(1+\alpha)}. \tag{3.3}$$

The result is sharp, the extremal function being of the form (3.2).

Proof. From Th. 2, we have

$$3^n(1+\alpha)\sum_{m=1}^{\infty}ma_m \le \sum_{m=1}^{\infty}(m+2)^n(m+\alpha)a_m \le (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty}ma_m \le \frac{1-\alpha}{3^n(1+\alpha)}.$$

Consequently, we obtain

$$|f'(z)| \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m r^{m-1} \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m \leq \frac{1}{r^2} + \frac{1-\alpha}{3^n(1+\alpha)}.$$

Also

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m r^{m-1} \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m \geq \frac{1}{r^2} - \frac{1-\alpha}{3^n(1+\alpha)}.$$

This completes the proof. ◊

Putting $n = 0$ in Th. 4, we get

Corollary 2. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,0}(\alpha) = \sigma^*_A(\alpha)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{1-\alpha}{1+\alpha} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{1+\alpha}.$$

The result is sharp.

We observe that our result in Cor. 2 improves the result of Uralegaddi and Ganigi [3, Th. 3 (Equation 4)].

4. Class preserving integral operators

In this section we consider the class preserving integral operators of the form (1.6).

Theorem 5. *Let the function $f(z)$ be defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then*

$$F(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{c}{c+m+1} a_m z^m, \quad c > 0$$

belongs to the class $\sigma^*_{A,n}(\beta(\alpha, n, c))$, where

$$\beta(\alpha, n, c) = \frac{(1+\alpha)(c+2) - c(1-\alpha)}{(1+\alpha)(c+2) + c(1-\alpha)}.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{3^n(1+\alpha)}z.$$

Proof. Suppose $f(z) \in \sigma^*_{A,n}(\alpha)$, then

$$\sum_{m=1}^{\infty}(m+2)^n(m+\alpha)a_m \leq (1-\alpha).$$

In view of Th. 2 we shall find the largest value of β for which

$$\sum_{m=1}^{\infty}\frac{(m+2)^n(m+\beta)}{(1-\beta)}\cdot\frac{c}{c+m+1}a_m \leq 1.$$

It suffices to find the range of values of β for which

$$\frac{c(m+2)^n(m+\beta)}{(1-\beta)(c+m+1)} \leq \frac{(m+2)^n(m+\alpha)}{(1-\alpha)} \quad \text{for each } m.$$

Solving the above inequality for β we obtain

$$\beta \leq \frac{(m+\alpha)(c+m+1)-mc(1-\alpha)}{(m+\alpha)(c+m+1)+c(1-\alpha)}.$$

For each α and c fixed let

$$F(m) = \frac{(m+\alpha)(c+m+1)-mc(1-\alpha)}{(m+\alpha)(c+m+1)+c(1-\alpha)}.$$

Then

$$F(m+1)-F(m) = \frac{A}{B} > 0 \quad \text{for each } m,$$

where

$$A = c(m+1)(m+2)(1-\alpha)$$

and

$$B = [(m+1+\alpha)(c+m+2)+c(1-\alpha)][(m+\alpha)(c+m+1)+c(1-\alpha)].$$

Hence $F(m)$ is an increasing function of m. Since

$$F(1) = \frac{(1+\alpha)(c+2)-c(1-\alpha)}{(1+\alpha)(c+2)+c(1-\alpha)}$$

the result follows. $\Diamond$

Remark. Putting $n = 0$ in the above theorems, we have the results obtained by Uralegaddi and Ganigi [3].

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