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A SANDWICH WITH CONVEXITY

Karol **Baron**

Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, PL–40–007 Katowice, Poland

Janusz **Matkowski**

Katedra Matematyki, Politechnika Łódzka, Filia w Bielsku-Białej, ul. Willowa 2, PL–43–309 Bielsko-Biała, Poland

Kazimierz **Nikodem**

Katedra Matematyki, Politechnika Łódzka, Filia w Bielsku-Białej, ul. Willowa 2, PL–43–309 Bielsko-Biała, Poland

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Abstract: We prove that real functions f and g, defined on a real interval I, satisfy

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

for all $x, y \in I$ and $t \in [0,1]$ iff there exists a convex function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$. Using this sandwich theorem we characterize solutions of two functional inequalities connected with convex functions and we obtain also the classical one-dimensional Hyers-Ulam Theorem on approximately convex functions.

Introduction

It is the aim of this note to characterize real functions which can be separated by a convex function. This leads us to functional inequality

$$(1)\qquad f(tx+(1-t)y) \le tg(x)+(1-t)g(y).$$

Using this characterization we describe also solutions of the inequalities

$$(2)\qquad f(tx+(T-t)y) \le tf(x)+(T-t)f(y)$$

and

$$(3)\quad f(tx+(T-t)y+(1-T)z_0) \le tf(x)+(T-t)f(y)+(1-T)f(z_0).$$

Functions fulfilling (2) appear in a connection with the converse of Minkowski's inequality in the case where the measure of the space considered is less than 1 (see [4; pp. 671–672] and [5; Remark 16]).

1. A sandwich theorem

Our main result reads as follows.

Theorem 1. *Real functions f and g, defined on a real interval I, satisfy (1) for all $x, y \in I$ and $t \in [0,1]$ iff there exists a convex function $h : I \to \mathbb{R}$ such that*

$$(4)\qquad f \le h \le g.$$

Proof. We argue as in [1; proof of Th. 2]. Assume that functions $f, g : I \to \mathbb{R}$ satisfy (1) and denote by E the convex hull of the epigraph of g:

$$E = \operatorname{conv}\{(x,y) \in I \times \mathbb{R} : g(x) \le y\}.$$

Let $(x,y) \in E$. It follows from the Carathéodory Theorem (see [3; Cor. 17.4.2] or [6; Th. 31E] or [7; the lemma on p. 88]) that (x,y) belongs to a two-dimensional simplex S with vertices in the epigraph of g. Denote

$$y_0 = \inf\{z \in \mathbb{R} : (x,z) \in S\}.$$

Then $y \ge y_0$ and (x, y_0) belongs to the boundary of S. Consequently $(x,y_0) = t(x_1,y_1) + (1-t)(x_2,y_2)$ with some $t \in [0,1]$ and (x_1,y_1), $(x_2,y_2) \in I \times \mathbb{R}$ such that $g(x_1) \le y_1$ and $g(x_2) \le y_2$. Hence, using also (1), we get

$$y \ge y_0 = ty_1 + (1-t)y_2 \ge tg(x_1) + (1-t)g(x_2) \ge$$
$$\ge f(tx_1 + (1-t)x_2) = f(x).$$

This allows us to define a function $h : I \to \mathbb{R}$ by the formula

$$h(x) = \inf\{y \in \mathbb{R} : (x,y) \in E\}$$

and gives $f \le h$. Moreover, since $(x, g(x)) \in E$ for every $x \in I$, we have also $h \le g$. It remains to show that h is convex. To this end fix arbitrarily $x_1, x_2 \in I$ and $t \in [0,1]$. Then, for any reals y_1, y_2 such that

$(x_1, y_1), (x_2, y_2) \in E$ we have $(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in E$, whence $h(tx_1 + (1-t)x_2) \le ty_1 + (1-t)y_2$. Passing to infimum we obtain the desired inequality: $h(tx_1+(1-t)x_2) \le th(x_1)+(1-t)h(x_2)$. This ends the proof (of the "only if" part but the "if" part is obvious). $\Diamond$

The following example shows that Th. 1 cannot be generalized for functions defined on a convex subset of the (complex) plane.

Example 1. Let $D \in \mathbb{C}$ be the open ball centered at zero and with the radius 2, and let z_1, z_2, z_3 be the (different) third roots of the unity. Define the functions f and g on D by the formulas

$$f(z) = \begin{cases} 0 & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases} \qquad g(z) = \begin{cases} 0 & \text{if } z \in \{z_1, z_2, z_3\} \\ 3 & \text{if } z \in D \setminus \{z_1, z_2, z_3\}. \end{cases}$$

It is easy to check that (1) holds for all $x, y \in D$ and $t \in [0,1]$. Suppose that there exists a convex function $h : D \to \mathbb{R}$ satisfying (4). Then

$$1 = f(0) = f\left(\frac{1}{3}(z_1 + z_2 + z_3)\right) \le h\left(\frac{1}{3}(z_1 + z_2 + z_3)\right) \le$$

$$\le \frac{1}{3}(h(z_1) + h(z_2) + h(z_3)) \le \frac{1}{3}(g(z_1) + g(z_2) + g(z_3)) = 0,$$

a contradiction.

Arguing as in the proof of Th. 1 we can get however the following results.

Theorem 1a. *Real functions f and g, defined on a convex subset D of an $(n-1)$-dimensional real vector space, satisfy*

$$f\left(\sum_{j=1}^{n} t_j x_j\right) \le \sum_{j=1}^{n} t_j g(x_j) \tag{5}$$

for all vectors $x_1, \ldots, x_n \in D$ and reals $t_1, \ldots, t_n \in [0,1]$ summing up to 1 iff there exists a convex function $h : D \to \mathbb{R}$ satisfying (4).

Theorem 1b. *Real functions f and g, defined on a convex subset D of a vector space, satisfy* (5) *for each positive integer n, vectors $x_1, \ldots, x_n \in$ $\in D$ and reals $t_1, \ldots, t_n \in [0,1]$ summing up to 1 iff there exists a convex function $h : D \to \mathbb{R}$ satisfying* (4).

2. Applications

We start with an application of Th. 1 connected with approximately convex functions.

If ε is a positive real number and a real function f, defined on a real interval I, satisfies

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon$$

for all $x, y \in I$ and $t \in [0,1]$, then (1) holds with $g = f + \varepsilon$ and it follows from Th. 1 that there exists a convex function $h : I \to \mathbb{R}$ such that

$$f(x) \le h(x) \le f(x) + \varepsilon \quad \text{for} \quad x \in I.$$

Putting $\varphi(x) = h(x) - \varepsilon/2$ we obtain a convex function $\varphi : I \to \mathbb{R}$ such that

$$|\varphi(x) - f(x)| \le \varepsilon/2 \quad \text{for} \quad x \in I.$$

This is the classical one-dimensional Hyers-Ulam Stability Theorem (see [2; Th. 2]; cf. also [1; Th. 2] and [3; Th. 17.4.2]).

Further applications of our Th. 1 concern solutions of the inequalities (2) and (3). Denote by J either $[0, +\infty)$ or $(0, +\infty)$. Given $T > 0$ and $f : J \to \mathbb{R}$ we define the function $f_T : J \to \mathbb{R}$ by the formula

$$f_T(x) = T^{-1} f(Tx).$$

Theorem 2. *Let T be a positive real number. A function $f : J \to \mathbb{R}$ satisfies (2) for all $x, y \in J$ and $t \in [0, T]$ iff there exists a convex function $\varphi : J \to \mathbb{R}$ such that*

$$\varphi_T \le f \le \varphi. \tag{6}$$

Proof. Assume that $f : J \to \mathbb{R}$ satisfies (2). Putting $T \cdot t$ in place of t in (2) we have

$$f_T(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{7}$$

for all $x, y \in J$ and $t \in [0,1]$. Applying Th. 1 we obtain a convex function $h : J \to \mathbb{R}$ such that

$$f_T \le h \le f. \tag{8}$$

Define now $\varphi : J \to \mathbb{R}$ by the formula

$$\varphi(x) = Th(T^{-1}x). \tag{9}$$

Then φ is convex and (6) holds.

Conversely, if (6) holds with a convex function $\varphi : J \to \mathbb{R}$ then (9) defines a convex function $h : J \to \mathbb{R}$ which satisfies (8) whence (7) follows for all $x, y \in J$ and $t \in [0,1]$. But this means that (2) holds for all $x, y \in J$ and $t \in [0, T]$. $\Diamond$

Example 2. If $T \in (0,1)$, then taking $\varphi(x) = x^2$ for $x \in [0, +\infty)$ we get by Th. 2 that every function $f : [0, +\infty) \to \mathbb{R}$ satisfying

$$Tx^2 \le f(x) \le x^2 \quad \text{for} \quad x \in [0, +\infty)$$

is a solution of (2). Similarly, if $T \in (1, +\infty)$, then taking $\varphi(x) = 1/x$ for $x \in (0, +\infty)$ we see that every function $f : (0, +\infty) \to \mathbb{R}$ such that

$$1/(T^2x) \leq f(x) \leq 1/x \quad \text{for} \quad x \in (0, +\infty)$$

satisfies (2).

Now we pass to inequality (3). Fix a real interval I and a point $z_0 \in I$. For $T \in (0,1)$ put

$$I_T^* = TI + (1-T)z_0 .$$

Given a real function φ with the domain containing I_T^*, we define $\varphi_T^* : I \to \mathbb{R}$ by the formula

$$\varphi_T^*(x) = T^{-1}(\varphi(Tx + (1-T)z_0) - (1-T)\varphi(z_0)).$$

Theorem 3. *Let $T \in (0,1)$. A function $f : I \to \mathbb{R}$ satisfies (3) for all $x, y \in I$ and $t \in [0,T]$ iff there exists a convex function $\varphi : I_T^* \to \mathbb{R}$ such that*

(10) $\varphi_T^*(x) \leq f(x)$ *for* $x \in I$ *and* $f(x) \leq \varphi(x)$ *for* $x \in I_T^*$.

Proof. Assume that f satisfies (3). Putting $T \cdot t$ in place of t in (3) we have

(11) $$f_T^*(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. Applying Th. 1 we obtain a convex function $h : I \to \mathbb{R}$ such that

(12) $$f_T^* \leq h \leq f .$$

Since $f_T^*(z_0) = f(z_0)$, we have $h(z_0) = f(z_0)$. Define $\varphi : I_T^* \to \mathbb{R}$ by the formula

(13) $$\varphi(x) = Th(T^{-1}(x - (1-T)z_0)) + (1-T)f(z_0).$$

Then φ is a convex function, $\varphi(z_0) = f(z_0)$,

$$\varphi_T^*(x) = h(x) \leq f(x) \quad \text{for} \quad x \in I$$

and

$$\varphi(x) \geq Tf_T^*(T^{-1}(x - (1-T)z_0)) + (1-T)f(z_0) = f(x) \quad \text{for} \quad x \in I_T^*.$$

Conversely, if (10) holds with a convex function $\varphi : I_T^* \to \mathbb{R}$ then $f(z_0) = \varphi(z_0)$ and (13) defines a convex function $h : I \to \mathbb{R}$ which satisfies (12). This implies (11) for all $x, y \in I$ and $t \in [0,1]$. Consequently f satisfies (3) for all $x, y \in I$ and $t \in [0,T]$. ◊

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